

Charged Rotating AdS Black Holes with Chern-Simons Coupling

Mozhgan Mir^{*,†,1} and Robert B. Mann^{*,2}

** Department of Physics and Astronomy, University of Waterloo,
Waterloo, Ontario, Canada, N2L 3G1*

*† School of Physics, Institute for Research in Fundamental Sciences (IPM),
P.O.Box 19395-5531, Tehran, Iran*

Abstract

We obtain a perturbative solution for rotating charged black holes in 5-dimensional Einstein-Maxwell-Chern-Simons theory with a negative cosmological constant. We start from a small undeformed Kerr-AdS solution and use the electric charge as a perturbative parameter to build up black holes with equal-magnitude angular momenta up to forth order. These black hole solutions are described by three parameters, the charge, horizon radius and horizon angular velocity. We determine the physical quantities of these black holes and study their dependence on the parameters of black holes and arbitrary Chern-Simons coefficient. In particular, for values of CS coupling constant beyond its supergravity amount, due to a rotational instability, counterrotating black holes arise. Also the rotating solutions appear to have vanishing angular momenta and do not manifest uniquely by their global charges.

¹mmir@ipm.ir

²rbmann@uwaterloo.ca

1 Introduction

Higher dimensional black holes have been subject of interest in recent years, particularly black hole solutions in asymptotically AdS spacetime due to their applications via the AdS/CFT conjecture to build the corresponding elements of the field theory [1] (for an application see [2] and references therein). The original higher-dimensional version of the Schwarzschild solution [3] was later extended to include rotation [4], and then further generalized to include a cosmological constant [5]. In D -dimensions, these solutions have $N = \lfloor \frac{D-1}{2} \rfloor$ independent spatial planes of rotation and N independent angular momenta.

One particular theory of interest is Einstein-Maxwell-Chern-Simons theory (EMCS) with a negative cosmological constant in five-dimensions. For a special value of the Chern-Simons (CS) coupling $\lambda = \lambda_{SG} = 1/(2\sqrt{3})$, this theory is equivalent to 5-dimensional minimal gauged supergravity [6]. However for arbitrary values of λ the exact rotating charged black hole solutions in five dimensions are not known. In the extremal case they have static horizons $\Omega_H = 0$ but nonzero angular momentum J [12].

In the absence of analytic solutions, one can follow either perturbative or numerical approaches. The closed form of rotating AdS EM black holes have been analyzed numerically [7, 8] and extended to include a Gauss-Bonnet term [9]. Numerical solutions for asymptotically flat EMCS black holes, in the case of spherical topology and two equal angular momenta, have also been found along with their extremal black hole counterparts [10, 11].

Such charged rotating black holes have a number of interesting features. For $\lambda_{SG} < \lambda < 2\lambda_{SG}$ the mass can decrease by increasing the angular momentum [10]. The extremal static black hole with zero Hawking temperature cannot emit energy and angular momentum via Hawking radiation; the angular momentum accumulates behind the horizon and causes rotational instability and a deformation of the black hole [13, 14]. In terms of the first law of thermodynamics, for extremal black holes with vanishing temperature and fixed charge ($dQ = 0$), this instability causes the horizon to rotate in the opposite direction of the angular momentum, a phenomenon called counterrotation [15]; since $dM = \Omega_H dJ$, an increase in J causes a decrease in M [13]. For $\lambda > 2\lambda_{SG}$, in addition to the static Reissner–Nordstrom solution, there are two extremal stationary solutions with vanishing global angular momenta [10]. The analysis of these cases is carried out by using a modified near-horizon formalism through inclusion of the CS term where the near horizon solutions are asymptotically flat [16, 17]. These black holes are not uniquely characterized by their global charges.

From a perturbative perspective, the gyromagnetic ratio for charged Kerr-AdS black holes was computed to leading order in the charge [18], and slowly rotating charged asymptotically flat black hole solutions were obtained by perturbing in terms of the angular momentum [19, 20]. Asymptotically flat charged rotating black holes in five dimensional Einstein-Maxwell and EMCS theories to higher order in the charge Q for equal-magnitude

of angular momenta have also been obtained [21, 22].

Here we construct general rotating AdS black holes in EMCS theory in 5 dimensions for arbitrary values of the CS coupling constant λ . We focus on black hole solutions with equal-magnitude angular momenta, employing the black hole charge as an expansion parameter. Related results using this approach for asymptotically flat extremal solutions have been obtained [22]. However our solutions form a three parameter family comprised of charge, horizon radius and horizon angular velocity, whose radii are small compared to the AdS_5 radius. The black hole uniqueness theorem in 4 dimensions states that asymptotically flat non-degenerate electrovac black holes are uniquely characterized by their global charges [23, 24, 25, 26]. We observe at order Q^3 in the charge that uniqueness is not satisfied, commensurate with previous work on EMCS theory [16]. Indeed we find that expanding to this order is sufficient to see many interesting features of these black holes, including various thresholds in the Chern-Simons couplings that introduce new behaviour in the various physical quantities of these black holes, counter-rotation perhaps being the most prominent example.

An outline to our paper is as follows. In section 2 we introduce the ansatz for the metric and the gauge potential for 5-dimensional black holes with equal-magnitude angular momenta. In section 3 we demonstrate the known analytic solution for λ_{SG} where this solution is rewritten in the coordinates given in [27]. Section 4 explains the perturbation method we employ for obtaining rotating charged black holes with arbitrary CS coefficient λ . In section 5 we present the expressions for the physical quantities of these rotating charged black holes: their energy, angular momentum, surface gravity and horizon area. Section 5.1 is devoted to analyze how these different physical quantities depend on the black hole parameters. Details of the field equations we employ with our ansatz and our perturbative solutions are given in appendices. For completeness we present the corresponding perturbative expansions from the exact known supergravity solution [6] in appendix C; we find our perturbative result fully consistent with this expansion to the order in which we work.

2 Basic equations of motion

The action for Einstein-Maxwell-Chern-Simons theory in 5 spacetime dimensions is

$$S = \frac{1}{16\pi} \int_{\mathcal{M}} d^5x \sqrt{-g} \left[R - 2\Lambda - F_{ab}F^{ab} + \frac{4}{3} \lambda \epsilon^{abcde} F_{ab}F_{cd}A_e \right] \quad (1)$$

where the cosmological constant $\Lambda = -6/l^2$ and the Chern-Simons coupling is λ . For $\lambda = \lambda_{SG}$ the action for five-dimensional gauged supergravity is recovered.

The field equations are given by

$$G_{ab} - \frac{6}{l^2} g_{ab} = 2T_{ab}, \quad \nabla_b F^{ba} + \lambda \epsilon^{abcde} F_{bc}F_{de} = 0 \quad (2)$$

where

$$T_{ab} = F_{ad}F_b{}^d - \frac{1}{4}g_{ab}F_{cd}F^{cd} \quad (3)$$

is the stress energy tensor for the gauge field.

We seek perturbative rotating black hole solutions to these equations by deploying the ansatz

$$ds^2 = -f(r)g(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \left[h(r) \left(d\psi + \frac{\cos\theta}{2}d\phi - \Omega(r)dt \right)^2 + \frac{1}{4}(d\theta^2 + \sin^2\theta d\phi^2) \right] \quad (4)$$

for the metric and

$$A(r) = a_0(r)dt + a_1(r) \left(\frac{1}{2}\cos(\theta)d\phi + d\psi \right) \quad (5)$$

for the gauge field. This ansatz, corresponding to a doubly-rotating black hole whose angular momenta are equal, ensures that the Einstein-AdS-Chern-Simons equations (2) reduce to a set of 6 ordinary differential equations for the 6 unknown functions $\{f, g, h, \Omega, a_0, a_1\}$.

In appendix A we present the field equations (2) that result upon using our ansatz (4,5) for the metric and gauge field. We are specifically interested in those solutions to (2) that asymptote to AdS₅ spacetime, *i.e.*, solutions whose large- r behaviour is given by

$$\begin{aligned} f|_{r \rightarrow \infty} &= \frac{r^2}{l^2} + 1 + \frac{C_f l^2}{r^2} + O(r^{-3}), & g|_{r \rightarrow \infty} &= 1 - \frac{C_h l^2}{r^2} + O(r^{-5}), \\ h|_{r \rightarrow \infty} &= 1 + \frac{C_h l^2}{r^2} + O(r^{-5}), & \Omega|_{r \rightarrow \infty} &= \frac{C_\Omega l^3}{r^4} + O(r^{-4}), \\ a_0|_{r \rightarrow \infty} &= \frac{C_{a_0} l^2}{r^2} + O(r^{-3}), & a_1|_{r \rightarrow \infty} &= \frac{C_{a_1} l^3}{r^2} + O(r^{-3}) \end{aligned} \quad (6)$$

For the black hole, the inner boundary is at its horizon where the function $f(r)$ must vanish. We take this condition as our definition for the location of the black hole horizon r_+ (the largest root of $f(r)$). The other functions must be regular at this hypersurface:

$$\begin{aligned} f|_{r \rightarrow r_+} &= \mathcal{O}(r - r_+), & g|_{r \rightarrow r_+} &= g(r_+) + \mathcal{O}(r - r_+), & h|_{r \rightarrow r_+} &= h(r_+) + \mathcal{O}(r - r_+) \\ \Omega|_{r \rightarrow r_+} &= \Omega_H + \mathcal{O}(r - r_+), & a_0|_{r \rightarrow r_+} &= a_0(r_+) + \mathcal{O}(r - r_+), & a_1|_{r \rightarrow r_+} &= a_1(r_+) + \mathcal{O}(r - r_+) \end{aligned} \quad (7)$$

These conditions fix some of constants of integration.

3 Exact Solution

One well-known solution to these equations is the metric [6]

$$\begin{aligned}
ds^2 &= -\frac{\Delta_{\bar{\theta}}[(1+g^2\bar{r}^2)\rho^2 dt + 2q\nu]dt}{\Xi_a\Xi_b\rho^2} + \frac{2q\nu\omega}{\rho^2} + \frac{f}{\rho^4} \left(\frac{\Delta_{\bar{\theta}}dt}{\Xi_a\Xi_b} - \omega \right)^2 + \frac{\rho^2}{\Delta_{\bar{r}}} d\bar{r}^2 + \frac{\rho^2}{\Delta_{\bar{\theta}}} d\bar{\theta}^2 \\
&\quad + \frac{\bar{r}^2 + a^2}{\Xi_a} \sin^2 \bar{\theta} d\bar{\phi}^2 + \frac{\bar{r}^2 + b^2}{\Xi_b} \cos^2 \bar{\theta} d\bar{\psi}^2 \\
A &= \frac{\sqrt{3}q}{\rho^2} \left(\frac{\Delta_{\bar{\theta}}dt}{\Xi_a\Xi_b} - \omega \right)
\end{aligned} \tag{8}$$

valid for $\lambda = \lambda_{SG}$, where

$$\begin{aligned}
\nu &= b \sin^2 \bar{\theta} d\bar{\phi} + a \cos^2 \bar{\theta} d\bar{\psi}, \\
\omega &= a \sin^2 \bar{\theta} \frac{d\bar{\phi}}{\Xi_a} + b \cos^2 \bar{\theta} \frac{d\bar{\psi}}{\Xi_b}, \\
\Delta_{\bar{\theta}} &= 1 - a^2 g^2 \cos^2 \bar{\theta} - b^2 g^2 \sin^2 \bar{\theta}, \\
\Delta_{\bar{r}} &= \frac{(\bar{r}^2 + a^2)(\bar{r}^2 + b^2)(1 + g^2 \bar{r}^2) + q^2 + 2abq}{\bar{r}^2} - 2m \\
\rho^2 &= \bar{r}^2 + a^2 \cos^2 \bar{\theta} + b^2 \sin^2 \bar{\theta}, \\
\Xi_a &= 1 - a^2 g^2, \quad \Xi_b = 1 - b^2 g^2, \\
f &= 2m\rho^2 - q^2 + 2abqg^2\rho^2
\end{aligned}$$

where g is related to the radius of AdS space $g = 1/l$ and q is related to the charge associated with the above gauge potential.

We are interested in finding charged doubly-rotating equal angular momenta solutions to the field equations, for which the rotation parameters $b = a = a_S$, and write $\Xi_S = 1 - \frac{a_S^2}{l^2}$. To do this we must express the solution (8) in terms of the coordinates given in (4). Using the coordinate transformation

$$r^2 = \frac{\bar{r}^2 + a_S^2}{\Xi_S} \quad \bar{\phi} = \psi - \phi/2 \quad \bar{\psi} = \psi + \phi/2$$

with $\bar{\theta} = \theta/2$, we obtain

$$\begin{aligned}
f(r) &= \frac{r^2}{l^2} + 1 - \frac{2m}{\Xi_S^2 r^2} + \frac{2Qa_S^2}{r^4 \Xi_S} + \frac{\Xi_S Q^2}{r^4} + \frac{2a_S^2 m}{\Xi_S^3 r^4} \\
h(r) &= \frac{1}{g(r)} = 1 + \frac{2Qa_S^2}{r^4 \Xi_S} - \frac{Q^2 a_S^2}{r^6} + \frac{2a_S^2 m}{\Xi_S^3 r^4} \\
a_S \Omega(r) &= 1 - \frac{r^4 + a_S^2 Q}{r^4 h(r)} = \frac{a_S^2}{r^6 \Xi_S^3 h(r)} \left(2mr^2 + \Xi_S Q r^2 \left(1 - \frac{a_S^4}{l^4} \right) - Q^2 \Xi_S^3 \right)
\end{aligned} \tag{9}$$

and where the gauge potential is

$$A = \frac{\sqrt{3}Q}{r^2} \left(dt - a_S(d\psi + \frac{\cos\theta}{2}d\phi) \right)$$

with $q = Q\Xi_S^2$.

Setting $a_S = 0$ we recover the Reissner-Nordstrom-AdS (RN-AdS) solution. Setting $Q = 0$ we recover the doubly-rotating Kerr-AdS (KAdS) metric [5]

$$\begin{aligned} f_0(r) &= 1 + \frac{r^2}{l^2} - \left(1 - \frac{a_M^2}{l^2} \right) \frac{r_M^2}{r^2} + \frac{a_M^2 r_M^2}{r^4}, & h_0(r) &= 1 + \frac{a_M^2 r_M^2}{r^4} \\ \Omega_0(r) &= \frac{a_M r_M^2}{r^4 h_0(r)}, & g_0(r) &= \frac{1}{h_0(r)} \end{aligned} \quad (10)$$

whose two angular momenta are equal, with $a_M = a_S|_{Q=0}$ and $r_M^2 = (2m/\Xi_S^3)|_{Q=0}$.

Note that we can rewrite r_M and a_M in terms of r_+ and Ω_H

$$r_M^2 = \frac{r_+^4 (l^2 + r_+^2)}{l^2 r_+^2 - a_M^2 (l^2 + r_+^2)}, \quad a_M = \frac{r_+^2 l^2 \Omega_H}{l^2 + r_+^2} \quad (11)$$

There are three free parameters for charged equally rotating black holes: the mass parameter m , the rotation parameter a and the charge parameter Q . We shall rewrite the former two quantities in terms of the horizon velocity Ω_H and the horizon radius r_+ . We can repeat a similar analysis for (9), from the condition that $f(r)$ defined in (8) vanishes at the event horizon $r = r_+$

$$m = -\frac{\Xi_{a_S}^2 (a_S^2 (2Q - g^2 r_+^4) + g^2 r_+^6 \Xi_{a_S} + Q^2 \Xi_{a_S}^2 + r_+^4)}{2(a_S^2 - r_+^2 \Xi_{a_S})} \equiv m_P \quad (12)$$

and the fact that $\Omega(r_+) = \Omega_H$ is a parameter independent of the charge Q and r_+ . This yields

$$a_S = \frac{2L^2 r_+^4 \Omega_H}{r_+^4 + L^2 (Q + r_+^2) + \sqrt{(L^2 (Q + r_+^2) + r_+^4)^2 - 4L^4 Q r_+^4 \Omega_H^2}} \equiv a_P \quad (13)$$

upon replacing m in (9), where we note that the above expression has a well-defined limit as $Q \rightarrow 0$.

Using (12) and (13) to replace a_S with a_P and m with m_P , all metric functions in (9) depend only on the parameters r_+ , Q , and Ω_H . We can expand the metric functions in any or all of these three parameters. Despite the presence of terms linear in Q in (9), we find upon replacement of a_S and m using (12) and (13) that no metric function has any terms

linear in Q – their series expansions begin with a Q^2 correction term to the metric functions in (10). We also find that the expansions begin with negative powers of r_+ ; this is consistent with the lack of an $r_+ \rightarrow 0$ limit as there are no boson stars in the setting we are considering.

No exact solutions to the Einstein-AdS-Chern-Simons system of equations (2,3) for general values of λ (including $\lambda = 0$) and non-zero charge Q are known. Here we aim to obtain such solutions perturbatively. We use the above explicit expansions to confirm validity of our solutions that we obtained by directly solving equations of motion for supergravity value of CS coupling.

4 Construction of the perturbative solution

We expand the metric, gauge field in a power series in a small parameter Q/l^2 that is related to the physical charge (see (22)). We then insert this expansion into the equations of motion, and expand latter in a power series in Q/l^2 , although it turns out that the resultant Maxwell equations do not seem to be analytically solvable. Therefore we use the matching method (described below) to produce the solutions.

To this end, the starting point of our setup is a KAdS black hole with arbitrary but small $r_+ \ll l$. We introduce two dimensionless perturbative parameters: one associated with the horizon length, r_+/l , and the other associated with the charge Q/l^2 of the gauge field. Hence we compute a three-parameter family of solutions, performing a double expansion in these parameters analogous to that carried out for a scalar field [27, 28]; our third parameter is the horizon angular velocity Ω_H .

The existence of the Chern-Simons term does not change the expression for the energy-momentum tensor, and so the expansion of this quantity begins at quadratic order in Q/l^2 . We therefore do not expect any terms linear in Q in the metric functions, consistent with the exact solution (9) upon replacement of a_S and m using (12) and (13).

Since the action remains invariant under the discrete symmetry

$$(Q, \lambda) \rightarrow (-Q, -\lambda) \tag{14}$$

changing only the sign of the charge (or alternatively the CS coupling) gives us another possible solution. The symmetry (14) imputes certain properties to the perturbative expansion. For instance, even (odd) powers of Q in the metric field's expansions appear with even (odd) powers of λ . However, gauge field functions change sign under this transformation, *i.e.*, even (odd) powers of Q accompany with odd (even) powers of λ . Therefore, we must take into account both odd and even powers in Q/l^2 due to the presence of Chern-Simons term [22, 29].

As mentioned before, this results in a set of ordinary differential equations that cannot be simultaneously solved everywhere in the spacetime, and so we shall employ a matched asymptotic expansion to deal with this situation. This involves dividing the exterior spacetime of the black hole into two regions: a near-region where $r_+ \leq r \ll l$ and a far-region where $r \gg r_+$. In each of these regions, we can discard terms in the equations with subdominant contributions. This yields in each region a coupled system of perturbative equations whose solutions are valid in different regions of approximation. Provided $r_+/l \ll 1$, the far and near regions have an overlapping zone, $r_+ \ll r \ll l$, and we then match the set of independent parameters that are produced in each of the two regions [27, 28, 30]. At each order in Q/l^2 we have a set of linear differential equations, which we solve subject to the requirements of the normalizability of functions at infinity together with their regularity at the horizon.

In the next section we describe details of the matching method. Then we present explicit results for our perturbative expansion in higher orders. We first present the structure of the perturbative ansatz we employ in both far and near field regions.

4.1 Far Field Region

Let us first focus on the region $r \gg r_+$. In this region the background (10) is a small perturbation about global AdS space. As discussed above, we start by performing a double expansion of our fields in $\epsilon = Q/l^2$ and r_+/l as follows:

$$F^{out}(r, Q, r_+) = F_0^{out}(r, r_+) + F_2^{out}(r, r_+)\epsilon^2 + \sum_{j=3}^n F_j^{out}(r, r_+)\epsilon^j \quad (15)$$

where

$$\begin{aligned} F_2^{out}(r, r_+) &= F_{2,-2}^{out}(r) \left(\frac{l^2}{r_+^2} \right) + F_{2,0}^{out}(r) + F_{2,2}^{out}(r) \left(\frac{r_+^2}{l^2} \right) + \dots \\ F_j^{out}(r, r_+) &= \sum_{i=-j}^p F_{j,2i+4}^{out}(r) \left(\frac{r_+}{l} \right)^{2i+4} \end{aligned} \quad (16)$$

and we have explicitly set $F_1^{out}(r, r_+) = 0$ since (upon inspection of the field equations) terms linear in Q are not present. The indices indicate the expansion order in Q and r_+ respectively.

For the gauge field

$$A^{out}(r, Q, r_+) = A_1^{out}(r, r_+)\epsilon + \sum_{j=2}^n A_j^{out}(r, r_+)\epsilon^j \quad (17)$$

with

$$A_1^{out}(r, r_+) = A_{1,0}^{out}(r) + A_{1,2}^{out}(r) \left(\frac{r_+^2}{l^2} \right) + \dots \quad A_j^{out}(r, r_+) = \sum_{i=-j}^p A_{j,2i+4}^{out}(r) \left(\frac{r_+}{l} \right)^{2i+4}$$

where $F^{out} = \{f^{out}, g^{out}, h^{out}, \Omega^{out}\}$ is shorthand for each of the metric functions in (4) in the far-region and A^{out} denotes either component of the gauge field. The occurrence of successively negative powers of r_+ with increasing powers of Q in (16) will be seen to be necessary once we apply the method to match the outer series expansion with the inner one that necessarily includes negative powers due to the form of the gauge field at leading order in Q . The perturbative expansion then proceeds similarly to the boson star cases except that instead of our background being global AdS we take it to be the KAdS black hole, whose metric functions for $Q = 0$ are denoted by $f_0(r), g_0(r), h_0(r)$ and $\Omega_0(r)$.

4.2 Near Field Region

For the near-field region $r_+ \leq r \ll l$, the metric (4) can be written as

$$ds^2 = \frac{r_+^2}{l^2} \left[-f g d\tau^2 + \frac{dz^2}{f} + z^2 \left[h \left(d\chi + \frac{\cos \theta}{2} d\phi - \frac{r_+}{l} \Omega d\tau \right)^2 + \frac{1}{4} (d\theta^2 + \sin^2 \theta d\phi^2) \right] \right] \quad (18)$$

using the radial coordinate $z = rl/r_+$ and $\tau = tl/r_+$. By applying the coordinate transformation, the rotation field emerges as $g_{\tau\psi} \propto \frac{r_+}{l} \Omega$. In the near region, as $r_+/l \ll 1$, it is suppressed relative to the other metric functions and can be constructed as a power series of r_+/l around static configuration. This argument remains valid when we add charge gradually to the system and have another power series expansion in small Q/l^2 .

For the KAdS metric, by applying this coordinate transformation on (10) we obtain

$$f = \frac{(L-z)(L+z)}{L^4 z^4} \left(\frac{L^6 (l^2 + r_+^2)}{l^2 (1 - r_+^2 \Omega_H^2) + r_+^2} - (l^2 + z^2) (L^4 + r_+^2 z^2) \right), \quad g = \frac{1}{h}$$

$$h = 1 - \frac{L^6 r_+^2 \Omega_H^2}{z^4 (l^2 (r_+^2 \Omega_H^2 - 1) - r_+^2)}, \quad \Omega = \frac{L^4 \Omega_H (l^2 + r_+^2)}{z^4 (l^2 + r_+^2) + l^2 r_+^2 \Omega_H^2 (L^4 - z^4)}$$

for the various metric functions, where we have replaced a_S and m using (12) and (13) with $Q = 0$.

The horizon is located at $z \simeq l$; hence $z/l \geq 1$, ensuring in turn that z/l is larger than $r_+/l \ll 1$. This yields a valid expansion in terms of the small horizon radius r_+ . Since the

dependence of the inner and outer solutions on r_+ differ in form, we extend our expansions to include negative powers of r_+ , writing for the metric functions

$$F^{in}(z, Q, r_+) = F_0^{in}(z, r_+) + \sum_{j=2}^n F_j^{in}(z, r_+) \left(\frac{Q}{l^2}\right)^j \quad F_j^{in}(z, r_+) = \sum_{i=-j}^p F_{j,2i}^{in}(z) \left(\frac{r_+}{l}\right)^{2i} \quad (19)$$

where we study the above series up to order four in charge (*i.e.*, $n = 4$). Based on the preceding discussion, we use a similar expansion $F^{in} = \{f^{in}, g^{in}, h^{in}, \Omega^{in}\}$ for the gravitational fields, where “in” denotes these series expansions are used only in the near-region. The near-region fields are obtained by solving the equations of motion (2) and imposing the horizon boundary conditions (7).

This situation likewise holds for the gauge field, and so the time component of the gauge field is given by

$$A_\tau^{in}(z, Q, r_+) = A_{\tau\,1}^{in}(z, r_+) \epsilon + \sum_{j=2}^n A_{\tau\,j}^{in}(z, r_+) \epsilon^j \quad (20)$$

where

$$\begin{aligned} A_{\tau\,1}^{in}(z, r_+) &= A_{\tau\,1,-1}^{in}(z) \left(\frac{l}{r_+}\right) + A_{\tau\,1,1}^{in}(z) \left(\frac{r_+}{l}\right) + A_{\tau\,1,3}^{in}(z) \left(\frac{r_+^3}{l^3}\right) + \dots \\ A_{\tau\,j}^{in}(z, r_+) &= \sum_{i=-j}^p A_{\tau\,j,2i+3}^{in}(r) \left(\frac{r_+}{l}\right)^{2i+3} \end{aligned}$$

written in terms of the τ coordinate. The angular component of the gauge field does not include the time coordinate and so

$$A_1^{in}(z, Q, r_+) = \sum_{j=1}^n A_{1\,j}^{in}(z, r_+) \epsilon^j \quad A_{1\,j}^{in}(z, r_+) = \sum_{i=-j}^p A_{1\,j,2i+2}^{in}(r) \left(\frac{r_+}{l}\right)^{2i+2} \quad (21)$$

retaining only even powers in r_+ . Note that in order to match the near-field with the far-field solutions the same time coordinate t must be used.

4.3 Matching Procedure

At leading order in Q/l^2 , ($n = 1$), we get a system of two linear differential equations from inserting the field expansion into the gauge field equations; it yields the expected $1/r^2$ behaviour for the zero component of the gauge field (chosen to be independent of r_+). The angular component of the gauge field has the same radial function to desired higher order solutions in r_+/l and decays as the boundary condition (6) dictates. At this order, gauge field does not react to the gravitational field and the background spacetime is the ($d = 5$)

KAdS black hole introduced in (10). From this analytic solution, the $n = 0$ coefficients F_0^{out} are known to any order $\mathcal{O}(r_+^p/l^p)$. One only needs to insert the expressions of rotations parameter a_M and mass-radius r_M of the KAdS black hole (10) to gravitational functions in order to get series expansion in r_+/l . These expansions satisfy the gravitational field equations (2) and the asymptotic boundary conditions (6) up to order $\mathcal{O}(Q/l^2, r_+^p/l^p)$, as expected.

The near-region analysis is similar. At leading order in Q , we again have a linear perturbation problem. The gravitational coefficients F_0^{in} are obtained from taking a series expansion in r_+/l of the KAdS background up to the desired order. Gauge field coefficients $A_{1,2i+2}^{in}$ satisfy (2) up to order $\mathcal{O}(Q/l^2, r_+^p/l^p)$ and boundary condition in (7).

4.3.1 Matching region

The far-region fields are valid for $r \gg r_+$ and the near ones for $r \ll l$. For small black holes we have $r_+ \ll l$. The far and near region solutions overlap where $r_+ \ll r \ll l$. Therefore this intersecting region implies matching conditions that fix the remaining integration constants to get solutions for the whole spacetime. The matching technique must be done at each order of r_+/l for a given order of Q/l^2 . In this procedure we match the near-region solution for large r with far-region solution as r becomes small. For this purpose, we take the Taylor expansion of (19), (20), (21) for large z and use the original radial coordinate $z = rl/r_+$, as well as the series expansion of (15), (17) for small- r . In employing this procedure, recall that for far-region solutions the coefficients of powers of r_+/r must be either removed or matched with the same terms in the near region. Furthermore, terms behaving like r/l in the inner region must either be eliminated or have counterparts in the far region solution.

Next we consider non-linear contributions $\mathcal{O}(Q^n/l^{2n}, r_+^p/l^p)$, ($n \geq 2$), where the field strength of the gauge field acts as a source in the Einstein equation, providing information about charged rotating black holes. We solve the coupled system of equations (2) order by order in (n, p) . From the charge corrections of the Maxwell solutions at order Q^n/l^{2n} , we proceed to Q^{n+1}/l^{2n+2} order to find the back-reaction of inserting extra charge on the gravitational fields.

We have solved the system of equations (2,3) using the ansatz (4,5) to order Q^4/l^8 , retaining powers of r_+ as relevant. We assume that $\mathcal{O}(Q/l^2) \sim \mathcal{O}(r_+/l)$. We managed to perform the perturbation method up to combined order $(n + p) \geq 5$ up to Q^3/l^6 . However at order Q^4/l^8 the calculations become extremely cumbersome, and we did not obtain high enough corrections in r_+/l , to be compatible with our lower order results. We therefore do not make use of them to obtain the physical quantities in section 5.1.

Upon solving the field equations, the arbitrary constants for the near-field solution that appear are fixed using the boundary conditions (7) at the horizon and by matching with the far-field solution. Likewise the constants in the far-field solution are fixed using the

boundary conditions (6). The results are very lengthy and so we present them in Appendix B. In this subsection we confine our remarks to some comments about our results. We find for both the near-field and far-field expansions that the metric and gauge fields exhibit $1/r^n$ radial behavior. This is due to the existence of the gauge field, and is as expected for a charged black hole. Up to order Q^3 , the solutions to the Einstein equations initially yield log and polylog terms that disappear in our final expressions after fixing constants from the boundary conditions; furthermore, and the log and polylog terms appearing in higher order are removed for $\lambda = \lambda_{SG}$. In general, to fix the constants of integration for $\Omega(r)$, we used the fact that Ω_H is the final value for the horizon angular velocity (without the necessity of including contributions from Q and r_+). It means the function $\Omega(r)$ is such that its corrections in Q vanish as $z \rightarrow l$. Only $\Omega(r)$ and the angular component of the gauge field depend on odd powers of Ω_H . When the sign of the azimuthal angular coordinate ϕ is changed, $\Omega_H \rightarrow -\Omega_H$; we will see in section 5.1 that this gives solutions with $J \rightarrow -J$. We note that a_0^{in} in our approximation does not receive Q^2 corrections (see(30)).

4.3.2 The Einstein-Maxwell Case

One useful by-product of our results is that for vanishing λ we also obtain expansions for charged rotating AdS black holes of equal angular momenta in Einstein-Maxwell theory. No analytic solutions are known for this case. Only even powers of Q in any metric function and only odd powers of Q in the gauge field components appear, and their explicit expressions can be obtained by setting $\lambda = 0$ in our results in the appendix. It is computationally much easier to go to even higher orders in Q but we have not included these expressions here.

5 Conserved Quantities and Thermodynamics

The metric (4) has five independent Killing vector fields: ∂_t , ∂_ψ and the three rotations of S^2 . However the presence of the gauge field yields a reduced level of symmetry, with the Killing vector given by

$$\zeta = \partial_t + \Omega_H \partial_\psi$$

That is null on and orthogonal to the event horizon $r = r_+$, which is therefore a Killing horizon. The horizon electrostatic potential Φ_H is given by

$$\Phi_H = (a_0 + \Omega_H a_1)|_{r=r_+}$$

The expression for finding the electric charge is the Gaussian integral

$$Q_E = \frac{1}{16\pi} \int_{S_\infty^3} (\star F - F \wedge A/\sqrt{3})$$

over the S^3 at infinity, where in 5 dimensions, the CS term will vanish when $r \rightarrow \infty$ and at the end only the non-zero term in a_0^{out} expansion in (29) gives a contribution.

We use the Ashtekar-Das formalism [31, 32] to calculate conserved quantities associated with the symmetries coming from asymptotic Killing vectors. For an asymptotic conformal Killing vector ξ , the corresponding conserved charge Q_ξ associated with this symmetry is

$$Q_\xi = \pm \frac{d-3}{8\pi} \int_\Sigma \tilde{\varepsilon}_{ab} \xi^a t^b d\Sigma,$$

where \pm denotes a timelike/spacelike conformal Killing vector, respectively, and $\tilde{\varepsilon}_{ab} = l^2 \tilde{K}_{abcd} n^b n^d$ is the electric part of the conformal Weyl tensor, where $\tilde{K}_{abcd} = \tilde{\Lambda}^{3-d} \tilde{C}_{abcd}|_{\tilde{\Lambda}=0}$ and $\tilde{\Lambda} = 1/r$ is the conformal factor. For $\xi = (\partial_t, \partial_\psi)$ we obtain

$$E = \frac{\pi l^2}{8} (4C_h - 3C_f), \quad J = \frac{\pi l^3 C_\Omega}{2}$$

for the energy and angular momentum respectively, where the constants C_f, C_h, C_Ω are given by the boundary conditions (6).

The temperature is specified by the surface gravity as $T_H = \sqrt{-(\nabla\zeta)^2}|_{r_+}/(2\sqrt{2}\pi)$ and the entropy is given by the area of black hole divided by 4. Hence

$$T_H = \frac{f' \sqrt{g}}{4\pi}|_{r_+}, \quad S = \frac{1}{4} A_{d-2} r_+^{d-2} \sqrt{h(r_+)}$$

where A_{d-2} is the area of $(d-2)$ -sphere.

Using the perturbative solutions in appendix B we find

$$Q_E = \frac{\sqrt{3}}{4} \pi l^2 \left(\frac{Q}{l^2} \right) \quad (22)$$

$$\begin{aligned} E = & \frac{\pi l^2}{4} \left[\frac{3}{2} \left(\frac{r_+^2}{l^2} \right) + \frac{3}{2} (l^2 \Omega_H^2 + 1) \left(\frac{r_+^4}{l^4} \right) + \frac{l^2 \Omega_H^2}{2} (3l^2 \Omega_H^2 + 1) \left(\frac{r_+^6}{l^6} \right) + \frac{l^2 \Omega_H^2}{2} (l^2 \Omega_H^2 - 1) \right. \\ & \times (3l^2 \Omega_H^2 + 1) \left(\frac{r_+^8}{l^8} \right) + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) + \left(\frac{Q^2}{l^4} \right) \left(\frac{3}{2} \left(\frac{l^2}{r_+^2} \right) - \frac{3l^4 \Omega_H^4}{2} \left(\frac{r_+^2}{l^2} \right) - \frac{l^2 \Omega_H^2}{2} (l^2 \right. \\ & \times \Omega_H^2 (3l^2 \Omega_H^2 - 8) - 1) \left(\frac{r_+^4}{l^4} \right) + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) \left. \right) + \left(\frac{Q^3}{l^6} \right) \left(6\sqrt{3} \lambda l^4 \Omega_H^4 - 2\sqrt{3} \lambda l^2 \Omega_H^2 (11 \right. \\ & \times l^2 \Omega_H^2 + 1) \left(\frac{r_+^2}{l^2} \right) + \mathcal{O} \left(\frac{r_+^4}{l^4} \right) \left. \right) + \left(\frac{Q^4}{l^8} \right) \left(\frac{1}{3} (1 - 12\lambda^2) l^2 \Omega_H^2 \left(\frac{l^4}{r_+^4} \right) + \mathcal{O} \left(\frac{l^2}{r_+^2} \right) \right) \left. \right] \end{aligned} \quad (23)$$

$$\begin{aligned}
J = & \frac{\pi l^3}{2} \left[l\Omega_H \left(\frac{r_+^4}{l^4} \right) + l^3\Omega_H^3 \left(\frac{r_+^6}{l^6} \right) + l^3\Omega_H^3 (l^2\Omega_H^2 - 1) \left(\frac{r_+^8}{l^8} \right) + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) + \left(\frac{Q^2}{l^4} \right) \right. \\
& \times \left(l\Omega_H - l\Omega_H (l^2\Omega_H^2 + 1) \left(\frac{r_+^2}{l^2} \right) - l\Omega_H (l^2\Omega_H^2 (l^2\Omega_H^2 - 3) - 1) \left(\frac{r_+^4}{l^4} \right) \right. \\
& + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) \left. \right) + \left(\frac{Q^3}{l^6} \right) \left(-2\sqrt{3}\lambda l\Omega_H \left(\frac{l^2}{r_+^2} \right) + 2\sqrt{3}\lambda l\Omega_H (3l^2\Omega_H^2 + 2) - 2\sqrt{3}\lambda l\Omega_H \right. \\
& \times (11l^2\Omega_H^2 + 3) \left(\frac{r_+^2}{l^2} \right) + \mathcal{O} \left(\frac{r_+^4}{l^4} \right) \left. \right) + \left(\frac{Q^4}{l^8} \right) \left(12\lambda^2 l\Omega_H \left(\frac{l^4}{r_+^4} \right) + \mathcal{O} \left(\frac{l^2}{r_+^2} \right) \right) \left. \right] \quad (24)
\end{aligned}$$

$$\begin{aligned}
S = & \frac{\pi^2 r_+^3}{2} \left[1 + \frac{l^2\Omega_H^2}{2} \left(\frac{r_+^2}{l^2} \right) + \frac{l^2\Omega_H^2}{8} (3l^2\Omega_H^2 - 4) \left(\frac{r_+^4}{l^4} \right) + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) + \left(\frac{Q^2}{l^4} \right) \left(-\frac{l^2\Omega_H^2}{2} \right. \right. \\
& \times \left(\frac{l^2}{r_+^2} \right) - \frac{l^2\Omega_H^2}{4} (l^2\Omega_H^2 - 6) + \mathcal{O} \left(\frac{r_+^2}{l^2} \right) \left. \right) + \left(\frac{Q^3}{l^6} \right) \left(2\sqrt{3}\lambda l^2\Omega_H^2 \left(\frac{l^4}{r_+^4} \right) - \sqrt{3}\lambda l^2 \right. \\
& \times \Omega_H^2 (l^2\Omega_H^2 + 8) \left(\frac{l^2}{r_+^2} \right) + \mathcal{O}(1) \left. \right) + \left(\frac{Q^4}{l^8} \right) \left(\mathcal{O} \left(\frac{l^6}{r_+^6} \right) \right) \left. \right] \quad (25)
\end{aligned}$$

$$\begin{aligned}
T_H = & \frac{1}{\pi r_+} \left[\frac{1}{2} + \left(1 - \frac{3l^2\Omega_H^2}{4} \right) \left(\frac{r_+^2}{l^2} \right) + \frac{l^2\Omega_H^2}{16} (4 - 5l^2\Omega_H^2) \left(\frac{r_+^4}{l^4} \right) - \frac{l^2\Omega_H^2}{32} (l^2\Omega_H^2 (7l^2\Omega_H^2 \right. \\
& - 16) + 8) \left(\frac{r_+^6}{l^6} \right) - \frac{l^2\Omega_H^2}{256} (l^2\Omega_H^2 (l^2\Omega_H^2 (45l^2\Omega_H^2 - 152) + 176) - 64) \left(\frac{r_+^8}{l^8} \right) \\
& + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) + \left(\frac{Q^2}{l^4} \right) \left(-\frac{1}{2} \left(\frac{l^4}{r_+^4} \right) + \frac{l^2\Omega_H^2}{2} \left(\frac{l^2}{r_+^2} \right) + \frac{l^2\Omega_H^2}{16} (3l^2\Omega_H^2 - 8) + \frac{l^2\Omega_H^2}{8} \right. \\
& \times (l^2\Omega_H^2 - 6) (l^2\Omega_H^2 - 1) \left(\frac{r_+^2}{l^2} \right) + \frac{5}{256} l^2\Omega_H^2 (l^2\Omega_H^2 (l^2\Omega_H^2 (5l^2\Omega_H^2 - 32) + 112) \\
& - 64) \left(\frac{r_+^4}{l^4} \right) + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) \left. \right) + \left(\frac{Q^3}{l^6} \right) \left(-\sqrt{3}\lambda l^2\Omega_H^2 \left(\frac{l^4}{r_+^4} \right) + \frac{\sqrt{3}}{2} \lambda l^2\Omega_H^2 (l^2\Omega_H^2 + 4) \right. \\
& \times \left(\frac{l^2}{r_+^2} \right) + \frac{\sqrt{3}}{8} \lambda l^2\Omega_H^2 (l^2\Omega_H^2 (l^2\Omega_H^2 + 4) - 32) + \frac{\sqrt{3}}{16} \lambda l^2\Omega_H^2 (l^2\Omega_H^2 (l^2\Omega_H^2 (l^2\Omega_H^2 \\
& - 24) - 72) + 128) \left(\frac{r_+^2}{l^2} \right) + \mathcal{O} \left(\frac{r_+^4}{l^4} \right) \left. \right) + \left(\frac{Q^4}{l^8} \right) \left(\frac{1}{18} (192\lambda^2 - 7) l^2\Omega_H^2 \left(\frac{l^6}{r_+^6} \right) \right. \\
& + \frac{1}{144} l^2\Omega_H^2 (-24\lambda^2 (133l^2\Omega_H^2 + 4\pi^2 + 146) + 77l^2\Omega_H^2 + 8\pi^2 + 76) \left(\frac{l^4}{r_+^4} \right) + \mathcal{O} \left(\frac{l^2}{r_+^2} \right) \left. \right) \left. \right] \quad (26)
\end{aligned}$$

$$\begin{aligned}
\Phi_H = & \frac{1}{r_+^2} \left[\left(\frac{Q}{l^2} \right) \left(\sqrt{3}l^2 \left(1 + l^2\Omega_H^2 \left(-\left(\frac{r_+^2}{l^2} \right) + \left(\frac{r_+^4}{l^4} \right) - \left(\frac{r_+^6}{l^6} \right) + \left(\frac{r_+^8}{l^8} \right) - \left(\frac{r_+^{10}}{l^{10}} \right) \right) \right) \right. \right. \\
& + \mathcal{O} \left(\frac{r_+^{12}}{l^{12}} \right) \left. \right) + \left(\frac{Q^2}{l^4} \right) \left(6\lambda l^4\Omega_H^2 - 6\lambda l^4\Omega_H^2 (l^2\Omega_H^2 + 2) \left(\frac{r_+^2}{l^2} \right) + 18\lambda l^4\Omega_H^2 (l^2\Omega_H^2 \right. \\
& + 1) \left(\frac{r_+^4}{l^4} \right) - 12\lambda l^4\Omega_H^2 (3l^2\Omega_H^2 + 2) \left(\frac{r_+^6}{l^6} \right) + \mathcal{O} \left(\frac{r_+^8}{l^8} \right) \left. \right) + \left(\frac{Q^3}{l^6} \right) \left(\frac{l^4\Omega_H^2}{\sqrt{3}} (1 \right.
\end{aligned}$$

$$\begin{aligned}
& -48\lambda^2) \left(\frac{l^2}{r_+^2} \right) + \frac{l^4 \Omega_H^2}{\sqrt{3}} (144\lambda^2 + (120\lambda^2 - 1) l^2 \Omega_H^2 - 3) - \frac{2l^4 \Omega_H^2}{\sqrt{3}} (12\lambda^2 (l^2 \Omega_H^2 + 6) \\
& \times (3l^2 \Omega_H^2 + 2) - 2l^2 \Omega_H^2 - 3) \left(\frac{r_+^2}{l^2} \right) + \mathcal{O} \left(\frac{r_+^4}{l^4} \right) \left(\frac{Q^4}{l^8} \right) \left(4\lambda (30\lambda^2 - 1) l^4 \Omega_H^2 \left(\frac{l^4}{r_+^4} \right) \right. \\
& \left. + 2\lambda l^4 \Omega_H^2 (-240\lambda^2 + (7 - 300\lambda^2) l^2 \Omega_H^2 + 8) \left(\frac{l^2}{r_+^2} \right) + \mathcal{O}(1) \right) \Big] \quad (27)
\end{aligned}$$

It is straightforward (but tedious) to check (up to the orders in our expansion that we have computed) that these quantities satisfy the first law of thermodynamics

$$dE = \Omega_H dJ + T_H dS + Q_E d\Phi_H$$

where the differentials can be computed by regarding J , S , and Φ_H as functions of the parameters Q , r_+ and Ω_H .

5.1 Further Exploration

In this section we study relationships between the various physical quantities associated with our solutions (rescaled by appropriate powers of l to be dimensionless) up to order Q^3/l^6 . We define $\alpha = 2\sqrt{3}\lambda$ so that $\alpha = 1$ is the supergravity solution (9). As noted above, for $\alpha \neq 0$, the $Q \rightarrow -Q$ symmetry of Einstein-Maxwell theory is broken and so we need to consider positive and negative charges separately.

In plotting our results we impose several physical constraints. First, we require non-negative values for E , S , and the temperature T . The effect of this is that $|\Omega_H|$ is bounded by some maximal value dependent on the other parameters in the solution. Furthermore, noting that (23) and (25) are polynomials in Ω_H , the requirement that $E > 0$ is easily satisfied for small Ω_H since the leading term is positive, whereas the largest power (Ω_H^6 to the order we are working) implies that $r_+^2 > Q$.

All plots are for quantities computed to order Q^3/l^6 unless otherwise stated. We can check the accuracy of our approximation by comparing it to the physical quantities associated with the exact solution (9) with $\alpha = 1$. The formulae for E and J for the supergravity solution are [6]

$$\begin{aligned}
E_{SG} &= \frac{\pi (4a_S^2 g^2 Q \Xi_S^2 - m(\Xi_S - 4))}{4\Xi_S^3} \\
J_{aSG} &= \frac{\pi a_S (Q \Xi_S^2 (a_S^2 g^2 + 1) + 2m)}{4\Xi_S^3}
\end{aligned}$$

where J_{aSG} denotes the angular momentum along one of two possible rotation planes in $d = 5$. Note that this is half the total angular momentum. After substituting different

parameters from (12) and (13) we find a value for Ω_H where E and J in above formulae diverge; explicitly it is

$$\Omega_{Hmax}^2 = \frac{r_+^2 + 1}{r_+^2}$$

and for larger values $\Omega_H > \Omega_{Hmax}$, E_{SG} becomes negative. For the specific choice of $r_+/l = .9$ the upper bound for $\Omega_H l$ is 1.49485. We illustrate this situation in figure 1.

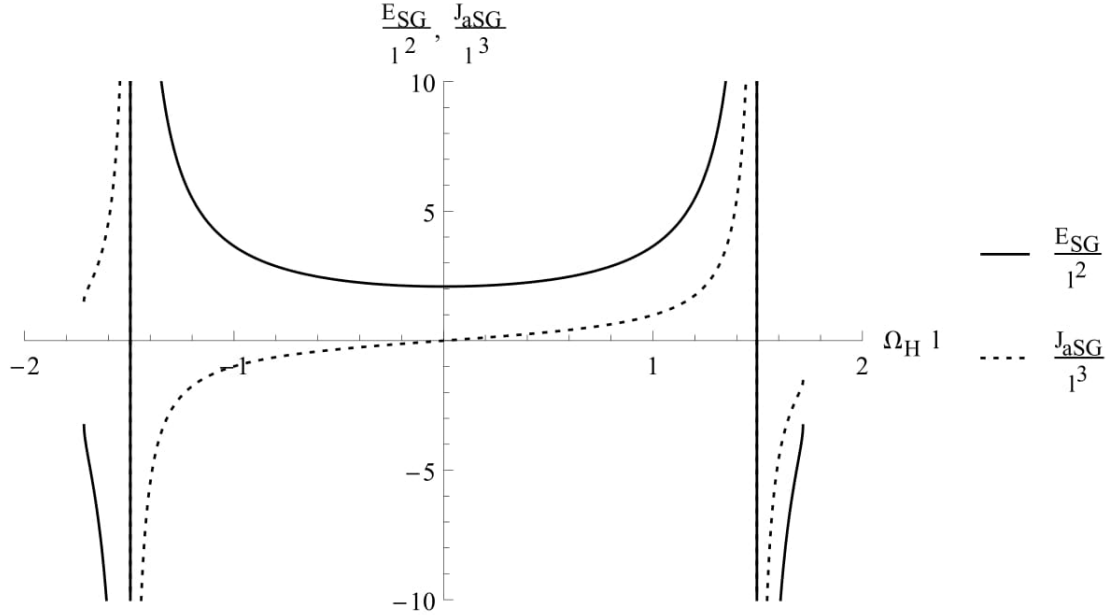


Figure 1: The exhibition of scaled E_{SG} and J_{aSG} in terms of $\Omega_H l$

We have numerically checked that there is a broad region of (Q, r_+, Ω_H) parameter space for which our results for the physical parameters of the black hole closely approximate those of the exact supergravity solution [6], for which $\alpha = 1$. All numerical values that appear in our plots are within this range. In subsequent plots we include the $\alpha = 1$ case for which exact results are known.

We plot in fig. 2 the energy (a) and entropy (b) as functions of Ω_H for fixed $Q = 0.5l^2$ and $r_+ = 0.9l$. The energy is initially an increasing function of Ω_H for all values of α . However at some value of $\alpha > 2$ the energy is maximized at some intermediate value $\Omega/l_H^2 < 1$, after which it is a decreasing function of Ω_H . The entropy is likewise an increasing function of Ω_H for $\alpha < 1$, but for larger values of α it monotonically decreases. The black hole shrinks in size as its horizon velocity gets larger, and for $\alpha > 2$ we find that the entropy vanishes for values of $\Omega/l_H^2 < 1$. For both E and S we find for increasing Q that the ‘fan’ of lines spreads out more. For smaller values of Q different curves overlap or change slowly.

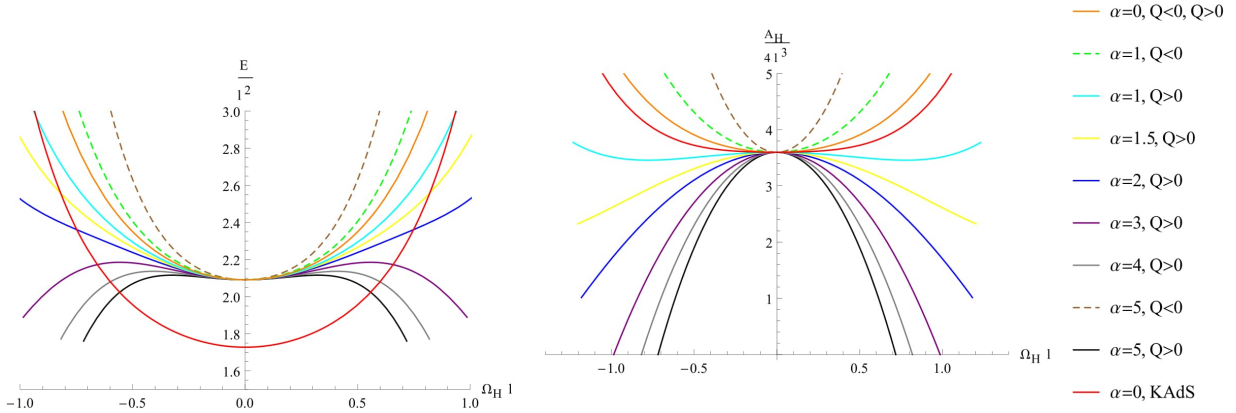


Figure 2: Energy (left) and entropy (right) versus horizon angular velocity for $Q/l^2 = 0.5$, $r_+/l = 0.9$, $\alpha = 0, \dots, 5$. When $\Omega_H = 0$ all curves intersect at $E/l^2 = 2.09082$ (a) and $A_H/4l^3 = 3.59747$ (b), which corresponds to the RN-AdS solution. For comparison the $Q = 0$, $\alpha = 0$ KAdS solution is also illustrated. Curves for positive and negative charges are presented by solid and dashed lines respectively.

We plot in Fig. 3 temperature (or surface gravity κ) as a function of Ω_H , retaining terms up to Q^4/l^8 . All curves end at an extremal solution with vanishing surface gravity and for all values of α , T is decreasing with horizon angular velocity. As α increases the magnitude of the negative slope becomes increasingly large with increasing Ω_H for both signs of Q , with the intersection point at extremality occurring for smaller values of Ω_H .

Figure 4 contains a plot of the angular momentum J as a function of the angular velocity of the horizon. For small values of $\alpha < 1$ the angular momentum increases with increasing Ω_H . Increasing α further, another threshold is encountered marking the onset of a rotational instability, above which there is a maximal value of J attainable for any Ω_H . For values of α above this second threshold, there appear for a given J two distinct solutions that have the same Q and r_+ but different horizon angular velocities. In particular there exist $J = 0$ solutions having the same Q and r_+ but differing angular velocities (and, we find, horizon areas). For even larger values of α , counterrotating black holes appear. These are black holes whose angular momenta are opposite in sign to the angular velocity of their horizon [15]. For the representative choice $\Omega_H l = 0.717879$, the critical values for α at the two thresholds mentioned above are respective 1.35182 and 1.93561; in general these depend on the value of Ω_H .

In figure 5 we plot horizon area (or entropy) as a function of angular momentum. We see that the horizon area increases with increasing J from its RN-AdS value at $J = 0$ for $\alpha < 1$. For $1 < \alpha < 3$ there is threshold above which the area is a decreasing function of increasing J . The black holes become more ‘squashed’ as J gets larger. We find that there are two possible values of the area for one value of J . We also find in this regime there exist black holes of the same area with two equal angular momenta with opposite signs.

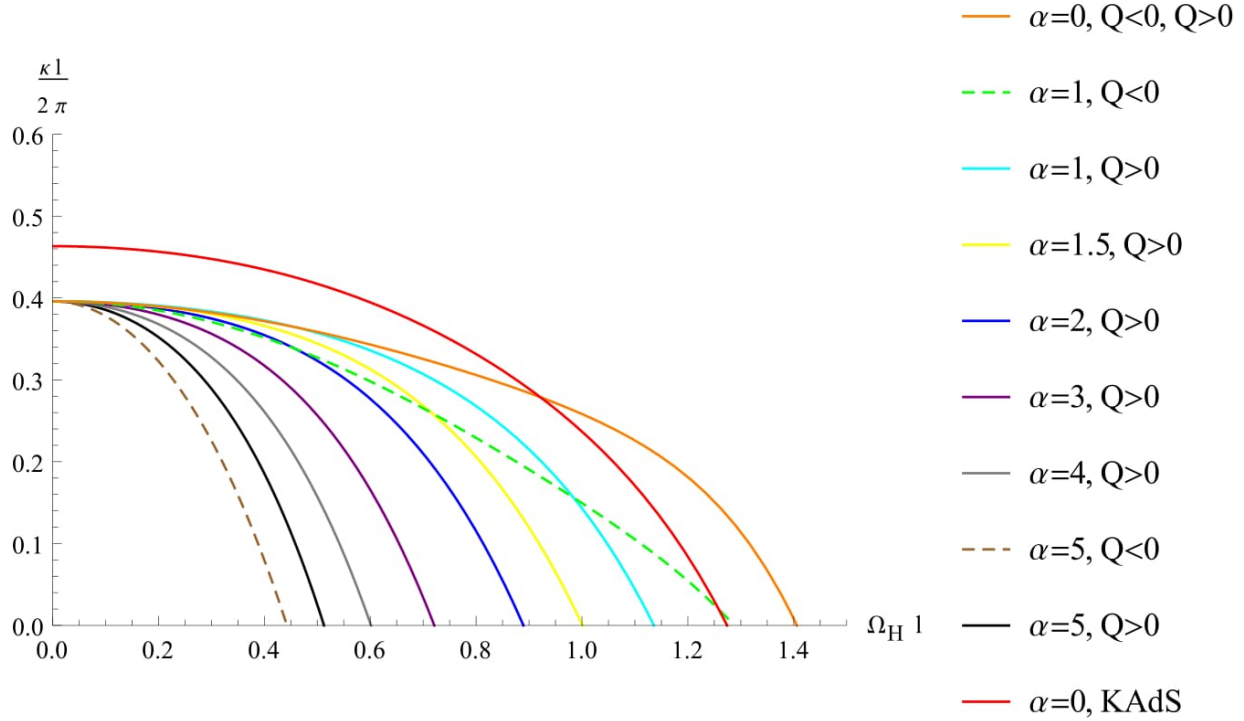


Figure 3: The temperature versus the horizon angular velocity. We choose $Q = 0.5l^2$, $r_+ = 0.9l$ and $\alpha = 0, \dots, 5$. The intersection point at $\Omega_H = 0$ is at 0.395935, corresponding to the RN-AdS solution. For comparison the $Q = 0$, $\alpha = 0$ KAdS solution is also illustrated. Curves for positive and negative charges are presented by solid and dashed lines respectively.

For the $\alpha \gtrsim 3$ we see from figure 5 that A_H goes to zero at some finite value of J . This phenomenon has been seen previously for extremal charged rotating black holes in Chern-Simons theory that are asymptotically flat, and is indicative of a rotational instability: for increasing α , instead of rotating faster, black holes start to quench [17].

The occurrence of solutions with zero total angular momentum is due to the cancellation between the horizon angular momentum and the angular momentum of the Maxwell field outside the horizon [10, 17]. For $\alpha < 2$, the $J = 0$ solutions must be static but for $\alpha \geq 2$ they are stationary (non-zero Ω_H) as appears in [33]. This is illustrated in fig 4. We also see that for sufficiently large α these solutions are not present.

The above results qualitatively hold for all small positive charges $Q > 0$. For $Q < 0$ all solutions have physical quantities whose qualitative dependence on Ω_H (or alternatively J) is similar to that for $Q > 0$ and $\alpha < 1$; for example for all $Q < 0$, horizon area increases as J increases. Comparing Figures in 4 we see that Ω_H is also well-behaved while A_H shrinks to zero.

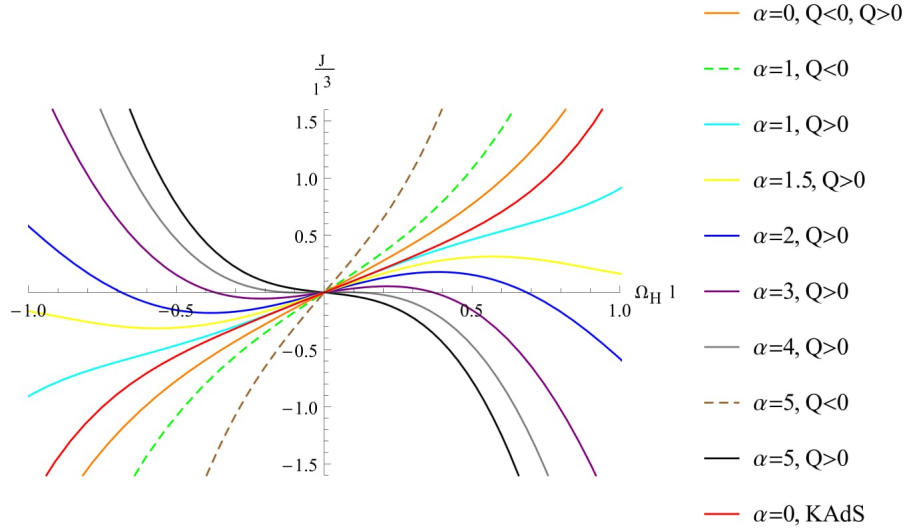


Figure 4: The angular momentum versus the horizon angular velocity. Curves for positive and negative charges are presented by solid and dashed lines respectively.

We also exhibit in figures 4 and 5 the effects of changing the signs of Ω_H and Q . For a given solution, changing $\Omega_H \rightarrow -\Omega_H$ yields a solution identical to one with the same parameters except for $J \rightarrow -J$. Counterrotation is illustrated quite clearly in Fig. 4 for $\alpha = 5$: we see that for $Q > 0$ the sign of Ω_H is always opposite to that of J for sufficiently large α . Indeed, (as noted previously) the only symmetry associated with changing $Q \rightarrow -Q$ is that of simultaneously changing $\lambda \rightarrow -\lambda$. Hence for a fixed λ solutions with $Q < 0$ behave very differently from those with $Q > 0$, as illustrated in fig. 4, which illustrates the $J \rightarrow -J$ symmetry, but not $Q \rightarrow -Q$, the distinction in the latter case growing more pronounced with increasing α .

To further illustrate the effects of the sign of Q we plot in fig. 6 energy as a function of the total angular momentum J for $Q > 0$ and $Q < 0$ for various values of α . For $Q < 0$ we see that E is an increasing function of J for all values of α . However for $Q > 0$ we see the same threshold values of α arise, with counter-rotation clearly visible for $\alpha = 4, 5$ (where the $J = 0$ solutions disappear) and double-valued behaviour for intermediate values $\alpha = 2, 3$.

We also see from figures 4 and 6 that the black hole solutions are not unique. As previously noted, there is a critical value α_{cr} above which there are $J = 0$ ($\Omega_H \neq 0$) solutions (see Fig.4). Since by changing $\Omega_H \rightarrow -\Omega_H$ the energy does not change, for fixed values of the charge there are two such $J = 0$ solutions.

In Fig. 6, as $\alpha = 0$ for both negative and positive values of charge we get the same curve that is no longer valid for non-zero CS coupling constant. We see that for large values of J the $\alpha = 0$ curve eventually tends to the KAdS curve.

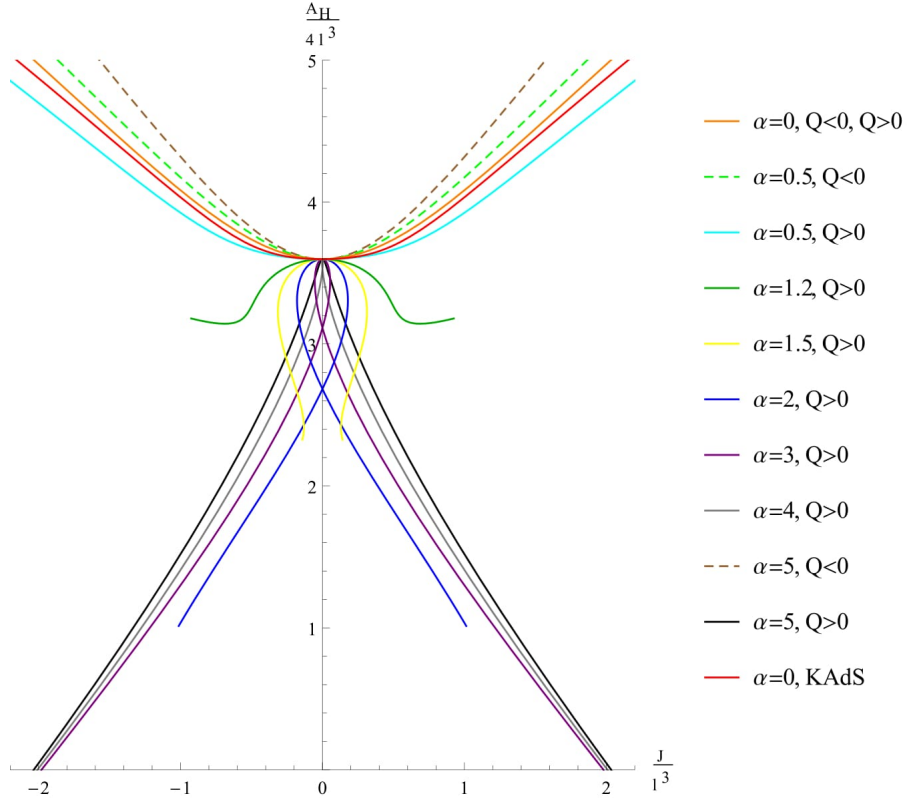


Figure 5: Horizon area versus the angular momentum. Curves for positive and negative charges are presented by solid and dashed lines respectively.

Next in figure 7 we plot energy, area (or entropy), angular momentum, and temperature (surface gravity) as a function of charge, the latter including Q^4/l^8 terms. We see that for $Q < 0$ all quantities monotonically increasing functions of $|Q|$ except for the surface gravity, which monotonically decreases. However for $Q > 0$ the situation is markedly different, with E, J increasing with increasing Q for small α but decreasing with increasing Q for large $\alpha > 1$. The counterrotation phenomenon is again evident with $J < 0$ above some threshold value of Q for sufficiently large α . Likewise there exists a maximal value of Q at which the area vanishes (fig. 7(b)) for large enough $\alpha > 2$. Note that since we chose $\Omega_H l = 0.717879$ for all graphs in order to make a comparison, in the given range of charge on the solution with the largest value of α ultimately attains zero horizon area.

In Fig. 7(d) (lower-right), we see for all values of α that T is a decreasing function of Q that reaches zero. Extremal black holes form the lower part of the boundary, since they have $\kappa = 0$. The uppermost curve is the RN-AdS solution, included for comparison.

In figure 8 we see for $Q > 0$ and $\alpha < 1$ the electrostatic potential is an initially increasing function of Q ; for larger values of α it begins to decrease eventually changing sign. For $Q < 0$, $|\Phi|$ increases with increasing $|Q|$. The right-hand part of shows that the electrostatic

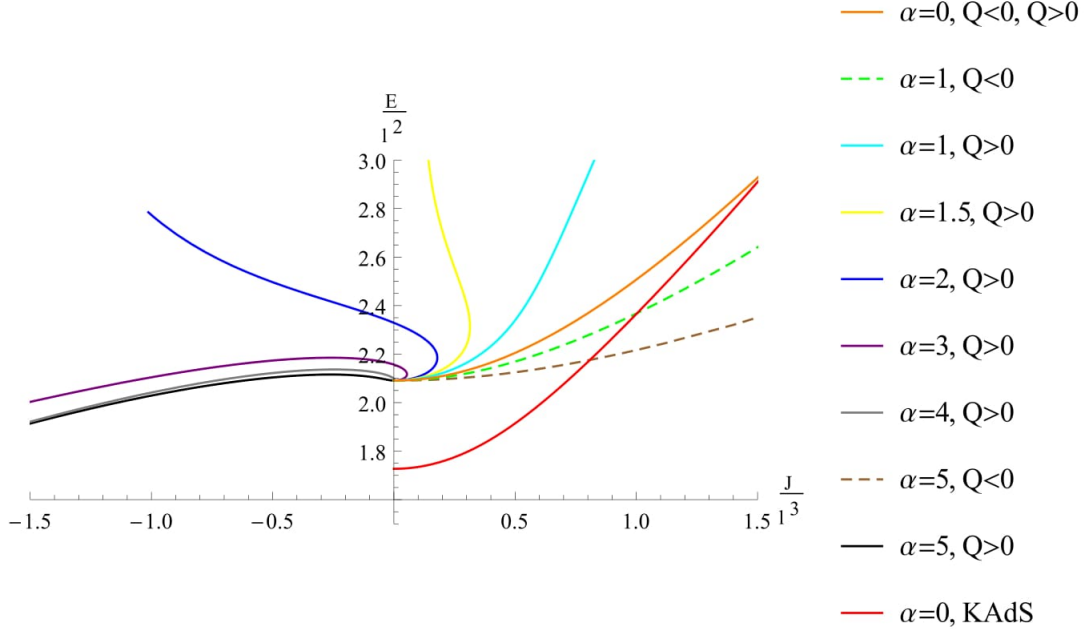


Figure 6: Energy versus the angular momentum for charge $|Q|/l^2 = 0.5$ and $r_+/l = 0.9$. The intersection with the energy axis at $E/l^2 = 2.09082$ for $J = 0$ is the RN-AdS solution. For completeness we include the $Q = 0$ KAdS solution. Curves for positive and negative charges are presented by solid and dashed lines respectively. These curves all reflect about the $J = 0$ axis for $\Omega_H < 0$; we have not drawn the associated graph.

potential is a decreasing function of Ω_H .

Finally in fig. 9 we illustrate the dependence of E and J on α . In Fig. 9 (a) for positive charge, with increasing α , E decreases whereas for negative charge it grows. For $\Omega_H l = 0.717879$ turns out that as $\alpha > 1.93561$, J becomes negative, indicative of counterrotation for $Q > 0$ that is absent for $Q < 0$.

6 Conclusion

We have obtained perturbative charged rotating AdS black hole solutions for Einstein-Maxwell-Chern-Simons theory for arbitrary values of the coupling $\alpha = \lambda/\lambda_{SG}$. The choice of equal magnitude angular momenta factorizes the angular dependence in the equations, yielding ordinary differential equations that are difficult but tractable. By perturbing about the Kerr-AdS solutions in 5 dimensions with electric charge Q and horizon radius r_+ as expansion parameters, we obtained series solutions in both the near and far regions. Our fourth order results are in agreement with the known supergravity solution presented in

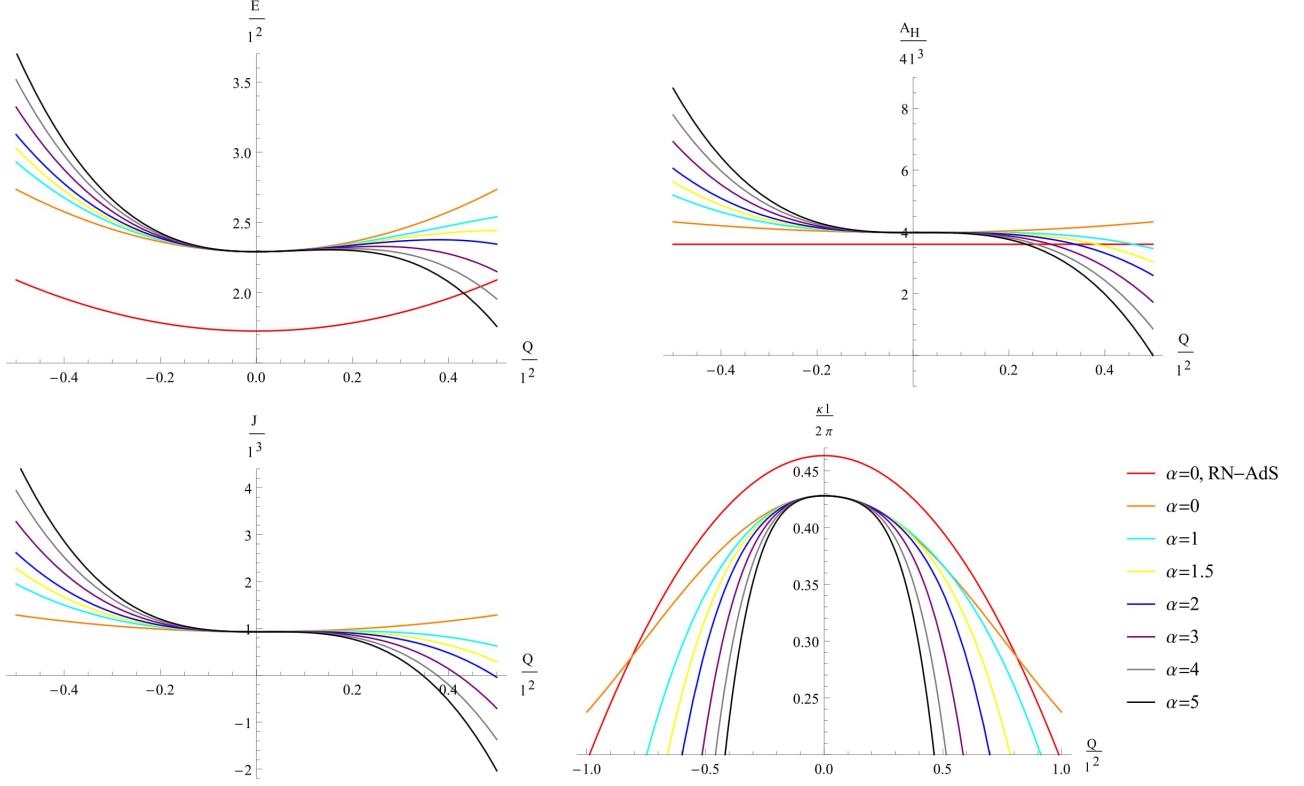


Figure 7: Energy (upper left), entropy (upper right), angular momentum (lower left), and temperature (lower right) plotted as a function of charge, with the $J = 0$ RN-AdS results plotted for comparison, with $r_+/l = 0.9$ and $\Omega_H l = 0.717879$. In the upper-left figure the intersection point at $Q = 0$ is at $E/l^2 = 2.29188$ corresponding to the KAdS solution for all values of α , whereas the corresponding point for the RN-AdS solution is at $E/l^2 = 1.72721$. In the remaining figures the $Q = 0$ intersection points are at $A_H/4l^3 = 3.97521$ (with the RN-AdS intersection at $A_H/4l^3 = 3.59747$) $J/l^3 = 0.927443$, and $\kappa l/2\pi = 0.360645$ (with the RN-AdS intersection at $\kappa l/2\pi = 0.463318$).

the Appendix C as $\alpha = 1$, verifying the validity of our perturbative approach. Our results also yield charged doubly-equally-rotating AdS black holes in Einstein-Maxwell theory by setting $\alpha = 0$.

Using these expansions we extract physical properties of the black holes. We find that interesting features emerge at cubic order in the charge, and observe that the behaviour of physical quantities depends strongly on the value of α and the sign of Q , qualitatively in agreement with numerical solutions previously obtained [10, 11]. Although $Q < 0$ solutions do not reveal any peculiar behaviour, the positive charge solutions present intriguing properties once the coupling constant exceeds its supergravity value ($\alpha > 1$).

Scaling all quantities in units of the AdS length, we observe the phenomenon of counter-

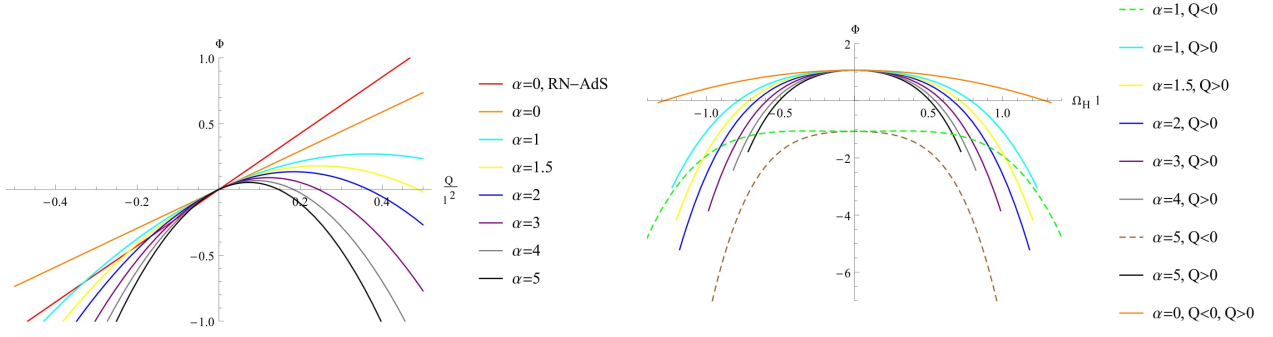


Figure 8: The electrostatic potential versus the charge (left) for fixed $\Omega_H l = 0.717879$ and the horizon angular velocity (right) for fixed $|Q|/l^2 = 0.5$. Curves for positive and negative charges are presented by solid and dashed lines respectively.

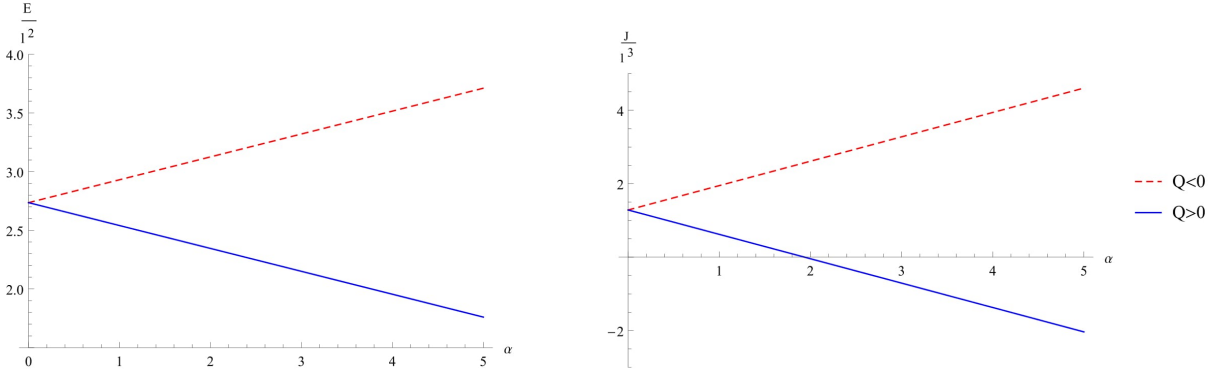


Figure 9: The energy (left) and the entropy (right) versus the CS coupling for $|Q|/l^2 = 0.5$, $r_+/l = 0.9$, $\Omega_H l = 0.717879$. In Fig. (a) the intersection of two lines is given by $E/l^2 = 2.73574$. In Fig. (b) The joining point is given by $J/l^3 = 1.28512$. Curves for positive and negative charges are presented by solid and dashed lines respectively.

rotation [15], with the signs of J and Ω_H opposite for sufficiently large α . For $\alpha > 1$, non-static $J = 0$ black holes exist, and upon setting $J \rightarrow -J$ we find two such stationary solutions. They have the same area horizon angular velocities of but opposite sign. Other physical charges remain the same, indicative of non-uniqueness [16]. Upon increasing α beyond its supergravity value, the horizon area starts to decrease with increasing J and eventually vanishes, signalling the onset of an instability [13, 14]. For large enough α , it goes to zero and the rotating $J = 0$ solutions disappear.

Although we worked in 5 dimensions, our procedure is applicable to arbitrary odd dimensional solutions in EMCS theory, when all the angular momenta are equal. We anticipate similar properties in a such theory upon including scalars.

Acknowledgements

MM would like to thank Francisco Navarro-Lerida for very useful comments and M.M. Sheikh-Jabbari for discussions; RBM would like to thank Jose Blasquez-Salcedo for interesting discussions as well. MM appreciates the hospitality of University of Waterloo during accomplishing this work. This work was supported in part by the Natural Sciences and Engineering Council of Canada.

Note added

During the preparation of this paper for publication, we received communication from J. Blasquez-Salcedo, J.Kunz, F. Navarro-Lerida, and E. Radu concerning their work [34] which overlaps with some results that we have obtained. Their approach is complementary to ours: they obtain numerical solutions to the EMCS equations whose asymptotic structure agrees with our solutions.

A Field Equations

Here we present the field equations and the gauge equations (2), simplified using our ansatz. The quantities C_1 and C_2 are the constraints that must also be satisfied. Our perturbative solutions to these equations appear in subsequent appendices.

$$\begin{aligned}
C_1 &\equiv g(r) \left(h(r) \left(2rf'(r) + 4f(r) - \frac{8r^2}{l^2} - 8 \right) + rf(r)h'(r) + 4h(r)^2 \right) + rf(r)h(r)g'(r) \\
&\quad + \frac{1}{3r^2} \left(r^4 h(r) (a'_0(r) + \Omega(r)a'_1(r))^2 - r^2 f(r)g(r)a'_1(r)^2 + 8a_1(r)^2 g(r)h(r) \right) = 0 \\
C_2 &\equiv \frac{1}{f(r)} \left(\frac{3f'(r)}{2r} - \frac{r^2 h(r)\Omega'(r)^2}{4g(r)} + \frac{h'(r)}{r} + \frac{5h(r)}{r^2} - \frac{6}{l^2} - \frac{8}{r^2} \right) + \frac{h'(r)}{f(r)h(r)} \left(\frac{f'(r)}{4} - \frac{2r}{l^2} \right. \\
&\quad \left. - \frac{2}{r} \right) + \frac{h'(r)}{h(r)} \left(\frac{h'(r)}{4h(r)} + \frac{3}{2r} \right) + \frac{3}{r^2} + \frac{1}{12r^4 f(r)g(r)h(r)^2} \left(r^2 (rh'(r) + 3h(r)) \left(r^2 h(r) \right. \right. \\
&\quad \left. \left. \times (a'_0(r) + \Omega(r)a'_1(r))^2 - f(r)g(r)a'_1(r)^2 \right) + 4a_1(r)^2 g(r)h(r) (2rh'(r) + 9h(r)) \right) = 0 \\
&\quad h''(r) + h'(r) \left(\frac{1}{f(r)} \left(-\frac{2h(r)}{r} + \frac{4r}{l^2} + \frac{4}{r} \right) + \frac{1}{r} \right) - \frac{h'(r)^2}{h(r)} + \frac{h(r)}{f(r)} \left(\frac{r^2 h(r)\Omega'(r)^2}{g(r)} \right. \\
&\quad \left. - \frac{8h(r)}{r^2} + \frac{8}{r^2} \right) - \frac{1}{6r^4 f(r)g(r)h(r)} \left(r^2 \left(r^3 h(r)h'(r) (a'_0(r) + \Omega(r)a'_1(r))^2 - f(r)g(r) \right. \right. \\
&\quad \left. \left. \times a'_1(r)^2 (rh'(r) + 6h(r)) \right) + 8a_1(r)^2 g(r)h(r) (rh'(r) + 3h(r)) \right) = 0
\end{aligned}$$

$$\begin{aligned}
& \Omega''(r) + \Omega'(r) \left(\frac{1}{f(r)} \left(f'(r) + \frac{2h(r)}{r} - \frac{4r}{l^2} - \frac{4}{r} \right) + \frac{2h'(r)}{h(r)} + \frac{7}{r} \right) + \frac{1}{6r^3 f(r) g(r) h(r)} \\
& \times \left(h(r) \Omega'(r) \left(r^4 (a'_0(r) + \Omega(r) a'_1(r))^2 + 8a_1(r)^2 g(r) \right) - r f(r) g(r) a'_1(r) (6a'_0(r) \right. \\
& \left. + a'_1(r) (r \Omega'(r) + 6\Omega(r))) \right) = 0 \\
& f''(r) - \frac{f'(r)^2}{f(r)} + f'(r) \left(\frac{1}{f(r)} \left(-\frac{6h(r)}{r} + \frac{12r}{l^2} + \frac{12}{r} \right) + \frac{1}{r} \right) + \frac{1}{f(r)} \left(8h(r) \left(-\frac{h(r)}{r^2} \right. \right. \\
& \left. \left. + \frac{4}{l^2} + \frac{4}{r^2} \right) - 32 \left(\frac{r^2}{l^4} + \frac{2}{l^2} + \frac{1}{r^2} \right) \right) + \frac{4h'(r)}{r} + \frac{8}{l^2} + \frac{8}{r^2} + \frac{1}{6l^2 r^4 f(r) g(r)^2 h(r)^2} \left(r^2 \right. \\
& \times \left(-l^2 r^3 f(r) h(r)^2 g'(r) (a'_0(r) + \Omega(r) a'_1(r))^2 + r g(r) h(r) \left(l^2 f(r) (r^2 h'(r) (a'_0(r) \right. \right. \right. \\
& \left. \left. + \Omega(r) a'_1(r))^2 - f(r) a'_1(r)^2 g'(r) \right) - 4l^2 r h(r)^2 (a'_0(r) + \Omega(r) a'_1(r))^2 + r h(r) (a'_0(r) \right. \\
& \left. + \Omega(r) a'_1(r)) \left(4l^2 r f(r) (a''_0(r) + a'_1(r) \Omega'(r)) + a'_0(r) (8(l^2 f(r) + l^2 + r^2) - l^2 r f'(r)) \right. \right. \\
& \left. \left. + \Omega(r) \left(4l^2 r f(r) a''_1(r) + a'_1(r) (8(l^2 f(r) + l^2 + r^2) - l^2 r f'(r)) \right) \right) \right) + f(r) g(r)^2 a'_1(r) \\
& \times \left(a'_1(r) \left(h(r) (l^2 (-rf'(r) + 4f(r) - 4h(r))) - 8(l^2 + r^2)) + l^2 r f(r) h'(r) \right) \right. \\
& \left. - 4l^2 r f(r) h(r) a''_1(r) \right) \Big) + 32l^2 r a_1(r) f(r) g(r)^2 h(r)^2 a'_1(r) + 8a_1(r)^2 g(r) h(r) \left(g(r) \left(h(r) \right. \right. \\
& \times \left(8(l^2 + r^2) - l^2 (rf'(r) + 4h(r)) \right) + l^2 r f(r) h'(r) \Big) + l^2 r f(r) h(r) g'(r) \Big) = 0 \\
& g''(r) - \frac{g'(r)^2}{g(r)} + g'(r) \left(\frac{1}{f(r)} \left(-\frac{4h(r)}{r} + \frac{8r}{l^2} + \frac{8}{r} \right) + \frac{1}{r} \right) + \frac{1}{f(r)} \left(g(r) \left(h(r) \left(-\frac{r^2 \Omega'(r)^2}{g(r)} \right. \right. \right. \\
& \left. \left. - \frac{8}{r^2} \right) + \left(\frac{4r}{l^2 h(r)} + \frac{4}{r h(r)} - \frac{6}{r} \right) h'(r) + \frac{8}{r^2} \right) \Big) - \frac{1}{6r^4 f(r) h(r)^2} \left(r^2 \left(h(r) (r^3 h'(r) (a'_0(r) \right. \right. \right. \\
& \left. \left. + \Omega(r) a'_1(r))^2 - 2r g(r) a'_1(r)^2 f'(r) - 2f(r) a'_1(r) (2r g(r) a''_1(r) + a'_1(r) (r g'(r) + g(r))) \right) \right. \\
& \left. + 4r^2 h(r)^2 (a'_0(r) + \Omega(r) a'_1(r)) (r a''_0(r) + 3a'_0(r) + r \Omega(r) a''_1(r) + r a'_1(r) \Omega'(r) + 3\Omega(r) \right. \\
& \left. \times a'_1(r)) + r f(r) g(r) a'_1(r)^2 h'(r) \right) + 32r a_1(r) g(r) h(r)^2 a'_1(r) + 8a_1(r)^2 h(r) (2r h(r) g'(r) \\
& \left. + g(r) (r h'(r) + h(r))) \right) = 0 \\
& a''_0(r) + a'_1(r) \left(a_1(r) \sqrt{g(r) h(r)} \left(\frac{8\lambda}{r^3 h(r)} - \frac{8\lambda \Omega(r)^2}{r f(r) g(r)} \right) - \frac{\Omega(r) f'(r)}{f(r)} + \frac{r^2 h(r) \Omega(r)^2 \Omega'(r)}{f(r) g(r)} \right. \\
& \left. - \frac{\Omega(r) g'(r)}{g(r)} + \frac{\Omega(r) h'(r)}{h(r)} + \Omega'(r) + \frac{2\Omega(r)}{r} \right) + \frac{4a_1(r) h(r) \Omega(r)}{r^2 f(r)} + a'_0(r) \left(-\frac{8\lambda a_1(r) \Omega(r)}{r f(r) g(r)} \right. \\
& \left. \times \sqrt{g(r) h(r)} + \frac{r^2 h(r) \Omega(r) \Omega'(r)}{f(r) g(r)} - \frac{g'(r)}{2g(r)} + \frac{h'(r)}{2h(r)} + \frac{3}{r} \right) = 0
\end{aligned}$$

$$\begin{aligned}
& a_1''(r) + \frac{h(r)}{r^2 f(r)} \left(a_1(r) \left(\frac{8\lambda r a_0'(r)}{\sqrt{g(r)h(r)}} - 4 \right) - \frac{r^4 a_0'(r) \Omega'(r)}{g(r)} \right) + a_1'(r) \left(\frac{8\lambda a_1(r) \Omega(r)}{r f(r) g(r)} \right. \\
& \times \sqrt{g(r)h(r)} + \frac{f'(r)}{f(r)} - \frac{r^2 h(r) \Omega(r) \Omega'(r)}{f(r) g(r)} + \frac{g'(r)}{2g(r)} - \frac{h'(r)}{2h(r)} + \frac{1}{r} \Big) = 0
\end{aligned}$$

B Perturbative results

Here we present our results for solving the field equations (2) perturbatively to order Q^4/l^8 , retaining relevant powers of r_+/l as appropriate. In the following, $\text{Li}_n(z)$ denotes the polylog function, defined as

$$\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \quad (28)$$

B.1 Far-field Expansion

In the far-field expansion we obtain

$$\begin{aligned}
f^{out}(r) &= 1 + \frac{r^2}{l^2} - \frac{l^2}{r^2} \left(\frac{r_+^2}{l^2} \right) - \frac{l^2 (l^2 \Omega_H^2 + 1)}{r^2} \left(\frac{r_+^4}{l^4} \right) + \frac{l^4 \Omega_H^2}{r^4} (l^2 (1 - \Omega_H^2 r^2) + r^2) \left(\frac{r_+^6}{l^6} \right) \\
&\quad - \frac{l^4 \Omega_H^2}{r^4} (l^2 \Omega_H^2 - 1) (l^2 (\Omega_H^2 r^2 - 1) - r^2) \left(\frac{r_+^8}{l^8} \right) + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) + \left(\frac{Q^2}{l^4} \right) \left(-\frac{l^2}{r^2} \right. \\
&\quad \times \left(\frac{l^2}{r_+^2} \right) + \frac{l^4}{r^4} + \frac{l^6 \Omega_H^4}{r^2} \left(\frac{r_+^2}{l^2} \right) + \frac{l^4 \Omega_H^2}{r^4} (r^2 (l^2 \Omega_H^2 (l^2 \Omega_H^2 - 4) + 1) - l^4 \Omega_H^2) \left(\frac{r_+^4}{l^4} \right) \\
&\quad \left. + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) \right) + \left(\frac{Q^3}{l^6} \right) \left(-\frac{4\sqrt{3}\lambda l^6 \Omega_H^4}{r^2} + \frac{4\sqrt{3}\lambda l^4 \Omega_H^2}{r^4} (l^4 \Omega_H^2 + r^2 (5l^2 \Omega_H^2 - 1)) \right. \\
&\quad \times \left(\frac{r_+^2}{l^2} \right) + \mathcal{O} \left(\frac{r_+^4}{l^4} \right) \Big) + \left(\frac{Q^4}{l^8} \right) \left(\frac{2(12\lambda^2 - 1) l^4 \Omega_H^2}{9r^2} \left(\frac{l^4}{r_+^4} \right) + \mathcal{O} \left(\frac{l^2}{r_+^2} \right) \right) \\
g^{out}(r) &= 1 - \frac{l^6 \Omega_H^2}{r^4} \left(\frac{r_+^6}{l^6} \right) - \frac{l^6 \Omega_H^2 (l^2 \Omega_H^2 - 1)}{r^4} \left(\frac{r_+^8}{l^8} \right) + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) + \left(\frac{Q^2}{l^4} \right) \left(\frac{l^6 \Omega_H^2}{r^6} (r^2 \right. \\
&\quad \times (l^2 \Omega_H^2 - 1) + l^2) \left(\frac{r_+^4}{l^4} \right) + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) \Big) - \left(\frac{Q^3}{l^6} \right) \left(\frac{4\sqrt{3}\lambda l^6 \Omega_H^2}{r^6} (r^2 (l^2 \Omega_H^2 - 1) \right. \\
&\quad \left. + l^2) \left(\frac{r_+^2}{l^2} \right) + \mathcal{O} \left(\frac{r_+^4}{l^4} \right) \right) + \left(\frac{Q^4}{l^8} \right) \left(\mathcal{O} \left(\frac{l^2}{r_+^2} \right) \right) \\
h^{out}(r) &= 1 + \frac{l^6 \Omega_H^2}{r^4} \left(\frac{r_+^6}{l^6} \right) + \frac{l^6 \Omega_H^2}{r^4} (l^2 \Omega_H^2 - 1) \left(\frac{r_+^8}{l^8} \right) + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) - \left(\frac{Q^2}{l^4} \right) \left(\frac{l^6 \Omega_H^2}{r^6} (r^2 \right.
\end{aligned}$$

$$\begin{aligned}
& \times (l^2 \Omega_H^2 - 1) + l^2 \left(\frac{r_+^4}{l^4} \right) + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) \Bigg) + \left(\frac{Q^3}{l^6} \right) \left(\frac{4\sqrt{3}\lambda l^6 \Omega_H^2}{r^6} (r^2 (l^2 \Omega_H^2 - 1) \right. \\
& \left. + l^2) \left(\frac{r_+^2}{l^2} \right) + \mathcal{O} \left(\frac{r_+^4}{l^4} \right) \right) + \left(\frac{Q^4}{l^8} \right) \left(\mathcal{O} \left(\frac{l^2}{r_+^2} \right) \right) \\
\Omega^{out}(r) = & \frac{l^4 \Omega_H}{r^4} \left(\frac{r_+^4}{l^4} \right) + \frac{l^6 \Omega_H^3}{r^4} \left(\frac{r_+^6}{l^6} \right) + \frac{l^6 \Omega_H^3 (l^2 \Omega_H^2 - 1)}{r^4} \left(\frac{r_+^8}{l^8} \right) + \left(\frac{Q^2}{l^4} \right) \left(\frac{l^4 \Omega_H}{r^4} - \frac{l^4 \Omega_H}{r^6} \right. \\
& \times (l^2 (\Omega_H^2 r^2 + 1) + r^2) \left(\frac{r_+^2}{l^2} \right) + \frac{l^4 \Omega_H}{r^6} (r^2 (1 - l^2 \Omega_H^2 (l^2 \Omega_H^2 - 3)) + l^2) \left(\frac{r_+^4}{l^4} \right) \\
& + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) \Bigg) + \left(\frac{Q^3}{l^6} \right) \left(- \frac{2\sqrt{3}\lambda l^4 \Omega_H}{r^4} \left(\frac{l^2}{r_+^2} \right) + \frac{2\sqrt{3}\lambda l^4 \Omega_H}{r^6} (r^2 (3l^2 \Omega_H^2 + 2) \right. \\
& \left. + l^2) - \frac{2\sqrt{3}\lambda l^4 \Omega_H}{r^6} (l^2 (\Omega_H^2 (l^2 + 11r^2) + 2) + 3r^2) \left(\frac{r_+^2}{l^2} \right) + \mathcal{O} \left(\frac{r_+^4}{l^4} \right) \right) \\
& + \left(\frac{Q^4}{l^8} \right) \left(\frac{12\lambda^2 l^4 \Omega_H}{r^4} \left(\frac{l^4}{r_+^4} \right) + \mathcal{O} \left(\frac{l^2}{r_+^2} \right) \right)
\end{aligned}$$

for the metric functions, and

$$\begin{aligned}
a_0^{out}(r) = & \left(\frac{Q}{l^2} \right) \left(\frac{\sqrt{3}l^2}{r^2} + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) \right) + \left(\frac{Q^2}{l^4} \right) \left(\mathcal{O} \left(\frac{r_+^6}{l^6} \right) \right) + \left(\frac{Q^3}{l^6} \right) \left(\mathcal{O} \left(\frac{r_+^2}{l^2} \right) \right) \\
& + \left(\frac{Q^4}{l^8} \right) \left(\mathcal{O} \left(\frac{l^2}{r_+^2} \right) \right) \tag{29} \\
a_1^{out}(r) = & \left(\frac{Q}{l^2} \right) \left(\frac{\sqrt{3}l^4 \Omega_H}{r^2} \left(- \left(\frac{r_+^2}{l^2} \right) + \left(\frac{r_+^4}{l^4} \right) - \left(\frac{r_+^6}{l^6} \right) + \left(\frac{r_+^8}{l^8} \right) \right) + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) \right) \\
& + \left(\frac{Q^2}{l^4} \right) \left(\frac{6\lambda l^4 \Omega_H}{r^2} - \frac{6\lambda l^4 \Omega_H}{r^2} (l^2 \Omega_H^2 + 2) \left(\frac{r_+^2}{l^2} \right) + \frac{18\lambda l^4 \Omega_H}{r^2} (l^2 \Omega_H^2 + 1) \left(\frac{r_+^4}{l^4} \right) \right. \\
& \left. + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) \right) + \left(\frac{Q^3}{l^6} \right) \left(- \frac{12\sqrt{3}\lambda^2 l^4 \Omega_H}{r^2} \left(\frac{l^2}{r_+^2} \right) + \frac{l^4 \Omega_H}{\sqrt{3}r^2} (108\lambda^2 + (96\lambda^2 + 1) \right. \\
& \left. \times l^2 \Omega_H^2) + \mathcal{O} \left(\frac{r_+^2}{l^2} \right) \right) + \left(\frac{Q^4}{l^8} \right) \left(\frac{72\lambda^3 l^4 \Omega_H}{r^2} \left(\frac{l^4}{r_+^4} \right) + \mathcal{O} \left(\frac{l^2}{r_+^2} \right) \right)
\end{aligned}$$

for the gauge field. It is straightforward to check that we recover the results in section C.1 upon setting $\lambda = \lambda_{SG}$.

B.2 Near-field Expansion

As for the exact case, the near-field expansions are considerably lengthier, and so we present the successive terms in the expansion for each function.

For the metric functions we find that

$$\begin{aligned}
f_0^{in}(z) &= 1 - \frac{l^2}{z^2} + \frac{1}{l^2 z^4} (l^8 \Omega_H^2 - l^4 z^2 (l^2 \Omega_H^2 + 1) + z^6) \left(\frac{r_+^2}{l^2} \right) + \frac{l^4 \Omega_H^2}{z^4} (l^2 \Omega_H^2 - 1) (l^2 - z^2) \\
&\quad \times \left(\frac{r_+^4}{l^4} \right) + \frac{l^4 \Omega_H^2}{z^4} (l^2 \Omega_H^2 - 1)^2 (l^2 - z^2) \left(\frac{r_+^6}{l^6} \right) + \frac{l^4 \Omega_H^2}{z^4} (l^2 \Omega_H^2 - 1)^3 (l^2 - z^2) \\
&\quad \times \left(\frac{r_+^8}{l^8} \right) + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) \\
f_2^{in}(z) &= \left(\frac{Q^2}{l^4} \right) \left(\frac{l^2 (l^2 - z^2)}{z^4} \left(\frac{l^4}{r_+^4} \right) - \frac{l^6 \Omega_H^4 (l^2 - z^2)}{z^4} - \frac{l^4 \Omega_H^2}{z^4} (l^2 \Omega_H^2 (l^2 \Omega_H^2 - 4) + 1) (l^2 - z^2) \right. \\
&\quad \times \left. \left(\frac{r_+^2}{l^2} \right) - \frac{l^4 \Omega_H^2}{z^4} (l^2 \Omega_H^2 (l^2 \Omega_H^2 (l^2 \Omega_H^2 - 5) + 10) - 3) (l^2 - z^2) \left(\frac{r_+^4}{l^4} \right) + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) \right) \\
f_3^{in}(z) &= \left(\frac{Q^3}{l^6} \right) \left(\frac{4\sqrt{3}\lambda l^6 \Omega_H^4}{z^4} (l^2 - z^2) \left(\frac{l^2}{r_+^2} \right) + \frac{4\sqrt{3}\lambda l^4 \Omega_H^2}{z^4} (1 - 5l^2 \Omega_H^2) (l^2 - z^2) \right. \\
&\quad \times \left. \frac{4\sqrt{3}\lambda l^4 \Omega_H^2}{z^4} (14l^2 \Omega_H^2 - 4) (l^2 - z^2) \left(\frac{r_+^2}{l^2} \right) + \mathcal{O} \left(\frac{r_+^4}{l^4} \right) \right) \\
f_4^{in}(z) &= \left(\frac{Q^4}{l^8} \right) \left(- \frac{(12\lambda^2 - 1) l^4 \Omega_H^2}{9z^8} (5l^6 - 6l^4 z^2 + 3l^2 z^4 - 2z^6) \left(\frac{l^6}{r_+^6} \right) + \frac{\Omega_H^2}{18z^{10}} (l^2 (6 \right. \\
&\quad \times (1 - 12\lambda^2) l^{12} \Omega_H^2 + 11 (12\lambda^2 - 1) l^{10} \Omega_H^2 z^2 - 6 (12\lambda^2 - 1) l^8 z^2 (\Omega_H^2 z^2 - 5) \\
&\quad - 6l^6 z^4 (96\lambda^2 + (96\lambda^2 + 1) \Omega_H^2 z^2 - 8) + l^4 z^6 (432\lambda^2 + (588\lambda^2 + 5) \Omega_H^2 z^2 - 36) \\
&\quad - 2 (81 + 2\pi^2) (12\lambda^2 - 1) l^2 z^8 + 144 (12\lambda^2 - 1) z^{10}) + 24 (12\lambda^2 - 1) z^8 \left((2l^4 \right. \\
&\quad - 9l^2 z^2 + 6z^4) \log \left(1 - \frac{l^2}{z^2} \right) - 6l^2 (l^2 - z^2) \log \left(\frac{z^2}{l^2} \right) - 24 (12\lambda^2 - 1) z^8 (2l^4 \\
&\quad - 9l^2 z^2 + 6z^4) \text{Li}_2 \left(\frac{l^2}{z^2} \right) \left. \right) \left(\frac{l^4}{r_+^4} \right) + \mathcal{O} \left(\frac{l^2}{r_+^2} \right) \\
g_0^{in}(z) &= 1 - \frac{l^6 \Omega_H^2}{z^4} \left(\frac{r_+^2}{l^2} \right) + \frac{l^6}{z^8} (l^6 \Omega_H^4 - \Omega_H^2 z^4 (l^2 \Omega_H^2 - 1)) \left(\frac{r_+^4}{l^4} \right) - \frac{l^6}{z^{12}} (l^6 \Omega_H^3 - \Omega_H z^4 \\
&\quad \times (l^2 \Omega_H^2 - 1))^2 \left(\frac{r_+^6}{l^6} \right) + \frac{l^6 \Omega_H^2}{z^{16}} (l^2 \Omega_H^2 (l^4 - z^4) + z^4)^3 \left(\frac{r_+^8}{l^8} \right) + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) \\
g_2^{in}(z) &= \left(\frac{Q^2}{l^4} \right) \left(\frac{l^8 \Omega_H^2}{z^6} \left(\frac{l^2}{r_+^2} \right) - \frac{l^6 \Omega_H^2}{z^{10}} (2l^2 z^4 + l^2 \Omega_H^2 (2l^6 - z^6) + z^6) + \frac{l^6 \Omega_H^2}{z^{14}} (3l^{14} \Omega_H^4 \right. \\
&\quad + z^{10} (l^2 \Omega_H^2 (l^2 \Omega_H^2 - 4) + 3) + 3l^2 z^8 - 2l^8 \Omega_H^2 z^4 (l^2 \Omega_H^2 - 3) - 2l^6 \Omega_H^2 z^6 (l^2 \Omega_H^2 \\
&\quad - 1)) \left(\frac{r_+^2}{l^2} \right) + \frac{l^6 \Omega_H^2}{z^{18}} (-4l^{20} \Omega_H^6 + z^{14} (l^2 \Omega_H^2 (l^2 \Omega_H^2 (l^2 \Omega_H^2 - 5) + 10) - 6) \\
&\quad - 4l^2 z^{12} + 6l^{14} \Omega_H^4 z^4 (l^2 \Omega_H^2 - 2) + 3l^{12} \Omega_H^4 z^6 (l^2 \Omega_H^2 - 1) - 2l^8 \Omega_H^2 z^8 (l^2 \Omega_H^2 \\
&\quad \times (l^2 \Omega_H^2 - 4) + 6) - 4l^6 \Omega_H^2 z^{10} (l^2 \Omega_H^2 (l^2 \Omega_H^2 - 3) + 2)) \left. \right) \left(\frac{r_+^4}{l^4} \right) + \mathcal{O} \left(\frac{r_+^6}{l^6} \right)
\end{aligned}$$

$$\begin{aligned}
g_3^{in}(z) &= \left(\frac{Q^3}{l^6}\right) \left(-\frac{4\sqrt{3}\lambda l^8 \Omega_H^2}{z^6} \left(\frac{l^4}{r_+^4}\right) + \frac{4\sqrt{3}\lambda l^6 \Omega_H^2}{z^{10}} (z^4 (3l^2 + z^2) + l^2 \Omega_H^2 (2l^6 + l^2 z^4 \right. \\
&\quad \left. - z^6)) \left(\frac{l^2}{r_+^2}\right) - \frac{4\sqrt{3}\lambda l^6 \Omega_H^2}{z^{14}} (2z^8 (3l^2 + 2z^2) + l^2 \Omega_H^2 (3l^{12} \Omega_H^2 + l^6 (8z^4 - 2\Omega_H^2 z^6) \right. \\
&\quad \left. + 2z^6 (l^4 + 2l^2 z^2 - 2z^4))\right) + \frac{4\sqrt{3}\lambda l^6 \Omega_H^2}{z^{18}} (10z^{12} (l^2 + z^2) + l^2 \Omega_H^2 (4l^{18} \Omega_H^4 + 20l^6 \\
&\quad \times z^8 + 10z^{12} (l^2 - z^2) - 3l^{12} \Omega_H^2 z^4 (l^2 \Omega_H^2 - 5) - 3l^{10} \Omega_H^2 z^6 (l^2 \Omega_H^2 - 1) + 2l^4 z^{10} \\
&\quad \times (l^2 \Omega_H^2 (l^2 \Omega_H^2 - 6) + 5))) \left(\frac{r_+^2}{l^2}\right) + \mathcal{O}\left(\frac{r_+^4}{l^4}\right) \\
g_4^{in}(z) &= \left(\frac{Q^4}{l^8}\right) \left(\frac{l^6 \Omega_H^2}{9z^6} (2(102\lambda^2 + 5) l^2 + 3(12\lambda^2 - 1) z^2) \left(\frac{l^6}{r_+^6}\right) + \frac{\Omega_H^2}{18z^{12} (l^2 - z^2)} \right. \\
&\quad \times \left(l^2 (l^2 - z^2) (18l^{14} \Omega_H^2 - 24(42\lambda^2 + 1) l^{12} \Omega_H^2 z^2 + 45(1 - 12\lambda^2) l^{10} \Omega_H^2 z^4 \right. \\
&\quad \left. - 2(732\lambda^2 + 11) l^8 \Omega_H^2 z^6 + 2l^6 z^6 (3(96\lambda^2 + 1) \Omega_H^2 z^2 - 8(102\lambda^2 + 5)) + 12 \right. \\
&\quad \times (1 - 66\lambda^2) l^4 z^8 + 12(1 - 12\lambda^2) l^2 z^{10} + 144(1 - 12\lambda^2) z^{12}) + 24(12\lambda^2 - 1) \\
&\quad \times z^{12} \left((l^4 - 7l^2 z^2 + 6z^4) \log\left(1 - \frac{l^2}{z^2}\right) - 2l^2(2l^2 - 3z^2) \log\left(\frac{z^2}{l^2}\right) - 24(12\lambda^2 \right. \\
&\quad \left. - 1) z^{12} (l^4 - 7l^2 z^2 + 6z^4) \text{Li}_2\left(\frac{l^2}{z^2}\right)\right) \left(\frac{l^4}{r_+^4}\right) + \mathcal{O}\left(\frac{l^2}{r_+^2}\right) \\
h_0^{in}(z) &= 1 + \frac{l^6 \Omega_H^2}{z^4} \left(\frac{r_+^2}{l^2}\right) + \frac{l^6 \Omega_H^2 (l^2 \Omega_H^2 - 1)}{z^4} \left(\frac{r_+^4}{l^4}\right) + \frac{l^6 \Omega_H^2 (l^2 \Omega_H^2 - 1)^2}{z^4} \left(\frac{r_+^6}{l^6}\right) \\
&\quad + \frac{l^6 \Omega_H^2 (l^2 \Omega_H^2 - 1)^3}{z^4} \left(\frac{r_+^8}{l^8}\right) + \mathcal{O}\left(\frac{r_+^{10}}{l^{10}}\right) \\
h_2^{in}(z) &= \left(\frac{Q^2}{l^4}\right) \left(-\frac{l^8 \Omega_H^2}{z^6} \left(\frac{l^2}{r_+^2}\right) + \frac{l^6 \Omega_H^2}{z^6} (l^2 (2 - \Omega_H^2 z^2) + z^2) + \frac{l^6 \Omega_H^2}{z^6} (-z^2 (l^2 \Omega_H^2 \right. \\
&\quad \times (l^2 \Omega_H^2 - 4) + 3) - 3l^2) \left(\frac{r_+^2}{l^2}\right) + \frac{l^6 \Omega_H^2}{z^6} (z^2 (6 - l^2 \Omega_H^2 (l^2 \Omega_H^2 (l^2 \Omega_H^2 - 5) + 10)) \\
&\quad \left. + 4l^2) \left(\frac{r_+^4}{l^4}\right) + \mathcal{O}\left(\frac{r_+^6}{l^6}\right)\right) \\
h_3^{in}(z) &= \left(\frac{Q^3}{l^6}\right) \left(\frac{4\sqrt{3}\lambda l^8 \Omega_H^2}{z^6} \left(\frac{l^4}{r_+^4}\right) - \frac{4\sqrt{3}\lambda l^6 \Omega_H^2}{z^6} (l^2 (\Omega_H^2 (l^2 - z^2) + 3) + z^2) \left(\frac{l^2}{r_+^2}\right) \right. \\
&\quad \left. + \frac{8\sqrt{3}\lambda l^6 \Omega_H^2}{z^6} (l^2 (2\Omega_H^2 (l^2 - z^2) + 3) + 2z^2) - \frac{40\sqrt{3}\lambda l^6 \Omega_H^2}{z^6} (l^2 (\Omega_H^2 (l^2 - z^2) \right. \\
&\quad \left. + 1) + z^2) \left(\frac{r_+^2}{l^2}\right) + \mathcal{O}\left(\frac{r_+^4}{l^4}\right)\right)
\end{aligned}$$

$$\begin{aligned}
h_4^{in}(z) &= \left(\frac{Q^4}{l^8} \right) \left(\frac{l^6 \Omega_H^2}{3z^6} (-12\lambda^2 (7l^2 + z^2) - 2l^2 + z^2) \left(\frac{l^6}{r_+^6} \right) + \frac{\Omega_H^2}{18z^{10}} \left(l^2 (8(12\lambda^2 - 1) \right. \right. \\
&\quad \times l^{12} \Omega_H^2 + 21(12\lambda^2 - 1) l^{10} \Omega_H^2 z^2 + 18(84\lambda^2 + 1) l^8 \Omega_H^2 z^4 - 6l^6 z^4 (48\lambda^2 (2\Omega_H^2 z^2 \\
&\quad - 7) + \Omega_H^2 z^2 - 8) + 12(66\lambda^2 - 1) l^4 z^6 + 36(12\lambda^2 - 1) l^2 z^8 + 144(1 - 12\lambda^2) \\
&\quad \times z^{10}) + 72(12\lambda^2 - 1) z^{10} \left((l^2 - 2z^2) \log \left(1 - \frac{l^2}{z^2} \right) - 2l^2 \right) \log \left(\frac{z^2}{l^2} \right) - 72 \\
&\quad \times (12\lambda^2 - 1) z^{10} (l^2 - 2z^2) \text{Li}_2 \left(\frac{l^2}{z^2} \right) \left(\frac{l^4}{r_+^4} \right) + \mathcal{O} \left(\frac{l^2}{r_+^2} \right) \Big) \\
\Omega_0^{in}(z) &= \frac{l^4 \Omega_H}{z^4} + \frac{l^6 \Omega_H^3}{z^8} (z^4 - l^4) \left(\frac{r_+^2}{l^2} \right) + \frac{l^6 \Omega_H^3}{z^{12}} (l^4 - z^4) (l^2 \Omega_H^2 (l^4 - z^4) + z^4) \left(\frac{r_+^4}{l^4} \right) \\
&\quad - \frac{l^6 \Omega_H^3}{z^{16}} (l^4 - z^4) (l^2 \Omega_H^2 (l^4 - z^4) + z^4)^2 \left(\frac{r_+^6}{l^6} \right) + \frac{l^6 \Omega_H^3}{z^{20}} (l^4 - z^4) (l^2 \Omega_H^2 (l^4 - z^4) \\
&\quad + z^4)^3 \left(\frac{r_+^8}{l^8} \right) + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) \\
\Omega_2^{in}(z) &= \left(\frac{Q^2}{l^4} \right) \left(-\frac{l^4 \Omega_H}{z^6} (l^2 - z^2) \left(\frac{l^4}{r_+^4} \right) + \frac{l^4 \Omega_H}{z^{10}} (l^2 - z^2) (l^2 \Omega_H^2 (2l^4 + l^2 z^2 + z^4) + z^4) \right. \\
&\quad \times \left(\frac{l^2}{r_+^2} \right) - \frac{l^4 \Omega_H}{z^{14}} (l^2 - z^2) (z^8 (1 - l^2 \Omega_H^2 (l^2 \Omega_H^2 - 3)) + l^4 \Omega_H^2 z^6 (3 - l^2 \Omega_H^2) + l^6 \\
&\quad \times \Omega_H^2 (l^4 \Omega_H^2 (3l^2 + 2z^2) + 4z^4) \Big) + \frac{l^4 \Omega_H}{z^{18}} (-z^{14} (l^2 \Omega_H^2 (l^2 \Omega_H^2 (l^2 \Omega_H^2 - 4) + 6) \\
&\quad + 1) + l^2 z^{12} + l^{18} \Omega_H^6 (4l^2 - z^2) + 3l^{14} \Omega_H^4 z^4 (3 - 2l^2 \Omega_H^2) - l^{12} \Omega_H^4 z^6 (l^2 \Omega_H^2 + 1) \\
&\quad + 2l^8 \Omega_H^2 z^8 (l^2 \Omega_H^2 (l^2 \Omega_H^2 - 3) + 3) + 3l^8 \Omega_H^4 z^{10} (l^2 \Omega_H^2 - 2) \Big) \left(\frac{r_+^2}{l^2} \right) + \frac{l^4 \Omega_H}{z^{22}} (z^{18} \\
&\quad \times (1 - l^2 \Omega_H^2 (l^2 \Omega_H^2 (l^2 \Omega_H^2 (l^2 \Omega_H^2 - 5) + 10) - 10)) - l^2 z^{16} + l^{24} \Omega_H^8 (z^2 - 5l^2) \\
&\quad + 4l^{20} \Omega_H^6 z^4 (3l^2 \Omega_H^2 - 4) + l^{18} \Omega_H^6 z^6 (l^2 \Omega_H^2 + 1) - 3l^{14} \Omega_H^4 z^8 (l^2 \Omega_H^2 (3l^2 \Omega_H^2 - 8) \\
&\quad + 6) + l^{12} \Omega_H^4 z^{10} (-l^2 \Omega_H^2 (6l^2 \Omega_H^2 - 11) - 2) + 2l^8 \Omega_H^2 z^{12} (l^2 \Omega_H^2 (l^2 \Omega_H^2 (l^2 \Omega_H^2 - 4) \\
&\quad + 6) - 4) + l^6 \Omega_H^2 z^{14} (l^2 \Omega_H^2 (l^2 \Omega_H^2 (5l^2 \Omega_H^2 - 17) + 18) - 2) \Big) \left(\frac{r_+^4}{l^4} \right) + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) \Big) \\
\Omega_3^{in}(z) &= \left(\frac{Q^3}{l^6} \right) \left(\frac{2\sqrt{3}\lambda l^4 \Omega_H}{z^6} (l^2 - z^2) \left(\frac{l^6}{r_+^6} \right) - \frac{2\sqrt{3}\lambda l^4 \Omega_H}{z^{10}} (l^2 - z^2) (z^4 (3l^2 \Omega_H^2 + 2) \right. \\
&\quad + l^4 \Omega_H^2 (3l^2 + 2z^2) \Big) \left(\frac{l^4}{r_+^4} \right) + \frac{2\sqrt{3}\lambda l^4 \Omega_H}{z^{14}} (l^2 - z^2) (z^8 (11l^2 \Omega_H^2 + 3) + l^6 \Omega_H^2 z^4 \\
&\quad \times (4l^2 \Omega_H^2 + 9) + l^4 \Omega_H^2 (l^6 \Omega_H^2 (5l^2 + 4z^2) + 8z^6) \Big) \left(\frac{l^2}{r_+^2} \right) - \frac{2\sqrt{3}\lambda l^4 \Omega_H}{z^{18}} (l^2 - z^2) \\
&\quad \times (20l^4 \Omega_H^2 z^{10} + 2z^{12} (13l^2 \Omega_H^2 + 2) + l^{16} \Omega_H^6 (7l^2 + 6z^2) + l^{12} \Omega_H^4 z^4 (l^2 \Omega_H^2 + 20) \\
&\quad - 4l^{10} \Omega_H^4 z^6 (l^2 \Omega_H^2 - 5) + 2l^6 \Omega_H^2 z^8 (9 - 2l^2 \Omega_H^2 (l^2 \Omega_H^2 - 5))) \Big) + \frac{2\sqrt{3}\lambda l^4 \Omega_H}{z^{22}} (l^2
\end{aligned}$$

$$\begin{aligned}
& -z^2) \left(40l^4\Omega_H^2 z^{14} + 5z^{16} (10l^2\Omega_H^2 + 1) + l^{22}\Omega_H^8 (9l^2 + 8z^2) + l^{18}\Omega_H^6 z^4 (35 - 6l^2 \right. \\
& \times \Omega_H^2) - 12l^{16}\Omega_H^6 z^6 (l^2\Omega_H^2 - 3) + l^{12}\Omega_H^4 z^8 (50 - l^2\Omega_H^2 (7l^2\Omega_H^2 - 11)) + 4l^{10}\Omega_H^4 \\
& \times z^{10} (l^2\Omega_H^2 (l^2\Omega_H^2 - 6) + 15) + 2l^6\Omega_H^2 z^{12} (2l^2\Omega_H^2 (l^2\Omega_H^2 (l^2\Omega_H^2 - 6) + 15) + 15) \Big) \\
& \times \left(\frac{r_+^2}{l^2} \right) + \mathcal{O} \left(\frac{r_+^4}{l^4} \right)
\end{aligned}$$

$$\begin{aligned}
\Omega_4^{in}(z) = & \left(\frac{Q^4}{l^8} \right) \left(\frac{12\lambda^2 l^4 \Omega_H}{z^6} (z^2 - l^2) \left(\frac{l^8}{r_+^8} \right) - \frac{l^4 \Omega_H}{9z^{12}} (l^2 - z^2) \left(9l^8 \Omega_H^2 - 2(204\lambda^2 + 1) l^6 \right. \right. \\
& \times \Omega_H^2 z^2 + 4(1 - 93\lambda^2) l^4 \Omega_H^2 z^4 + (1 - 660\lambda^2) l^2 \Omega_H^2 z^6 - 324\lambda^2 z^6 \Big) \left(\frac{l^6}{r_+^6} \right) \\
& + \frac{l^2 \Omega_H}{18z^{16}} \left(l^2 (54l^{16} \Omega_H^4 - 4(354\lambda^2 + 11) l^{14} \Omega_H^4 z^2 + 2(17 - 96\lambda^2) l^{12} \Omega_H^4 z^4 \right. \\
& + l^{10} \Omega_H^2 z^4 (54 - (1356\lambda^2 + 49) \Omega_H^2 z^2) + l^8 \Omega_H^2 z^6 ((1716\lambda^2 + 19) \Omega_H^2 z^2 \\
& - 4(816\lambda^2 + 13)) - 24l^6 \Omega_H^2 z^8 (6\lambda^2 (3\Omega_H^2 z^2 + 2) - 1) + 2l^4 \Omega_H^2 z^{10} (- 1428\lambda^2 \\
& + 7(120\lambda^2 - 1) \Omega_H^2 z^2 + 11) + 8l^2 z^{10} ((3(315 + 4\pi^2) \lambda^2 - \pi^2 - 18) \Omega_H^2 z^2 - 162\lambda^2) \\
& + 48z^{12} (27\lambda^2 + 2(1 - 12\lambda^2) \Omega_H^2 z^2) \Big) - 48(12\lambda^2 - 1) \Omega_H^2 z^{12} (z^2 - 2l^2) \left(2l^2 - (l^2 - 2z^2) \right. \\
& \times \log \left(1 - \frac{l^2}{z^2} \right) \Big) \log \left(\frac{z^2}{l^2} \right) + 48(12\lambda^2 - 1) \Omega_H^2 z^{12} (2l^4 - 5l^2 z^2 + 2z^4) \text{Li}_2 \left(\frac{l^2}{z^2} \right) \Big) \\
& \times \left(\frac{l^4}{r_+^4} \right) + \mathcal{O} \left(\frac{l^2}{r_+^2} \right)
\end{aligned}$$

and also

$$\begin{aligned}
a_{0\ 1}^{in}(z) &= \left(\frac{Q}{l^2} \right) \left(\frac{\sqrt{3}l^2}{z^2} \left(\frac{l}{r_+} \right) + \mathcal{O} \left(\frac{r_+^{11}}{l^{11}} \right) \right) \\
a_{0\ 2}^{in}(z) &= \left(\frac{Q^2}{l^4} \right) \left(\mathcal{O} \left(\frac{r_+^7}{l^7} \right) \right) \tag{30} \\
a_{0\ 3}^{in}(z) &= \left(\frac{Q^3}{l^6} \right) \left(- \frac{(12\lambda^2 - 1) l^8 \Omega_H^2}{\sqrt{3}z^6} \left(\frac{l^3}{r_+^3} \right) - \frac{(12\lambda^2 - 1) l^8 \Omega_H^2}{3\sqrt{3}z^8} (l^2 \Omega_H^2 (2l^2 - z^2) - 9z^2) \right. \\
& \times \left(\frac{l}{r_+} \right) + \frac{(12\lambda^2 - 1) l^4 \Omega_H^2}{6\sqrt{3}z^{10}} \left(- 3l^{12} \Omega_H^4 - 8l^2 \Omega_H^2 z^6 (l^2 - 4z^2) - 2l^8 \Omega_H^2 z^2 (l^2 \Omega_H^2 \right. \\
& - 10) + 2l^4 z^4 (l^2 \Omega_H^2 - 6) (l^2 \Omega_H^2 + 3) + 16\Omega_H^2 z^8 \left(2l^2 - (l^2 - 2z^2) \log \left(1 - \frac{l^2}{z^2} \right) \right) \\
& \times \log \left(\frac{z^2}{l^2} \right) + 16\Omega_H^2 z^8 (l^2 - 2z^2) \text{Li}_2 \left(\frac{l^2}{z^2} \right) \Big) \left(\frac{r_+}{l} \right) + \mathcal{O} \left(\frac{r_+^3}{l^3} \right) \Big)
\end{aligned}$$

$$\begin{aligned}
a_{0\ 4}^{in}(z) &= \left(\frac{Q^4}{l^8}\right) \left(\frac{4\lambda(12\lambda^2-1)l^8\Omega_H^2}{z^6} \left(\frac{l}{r_+}\right)^5 + \frac{8\lambda(12\lambda^2-1)l^8\Omega_H^2}{3z^8} (l^4\Omega_H^2 - 2z^2(l^2\Omega_H^2 \right. \\
&\quad \left. + 3)) \left(\frac{l^3}{r_+^3}\right) + \mathcal{O}\left(\frac{l}{r_+}\right)\right) \\
a_{1\ 1}^{in}(z) &= \left(\frac{Q}{l^2}\right) \left(\frac{\sqrt{3}l^4\Omega_H}{z^2} \left(-1 + \left(\frac{r_+^2}{l^2}\right) - \left(\frac{r_+^4}{l^4}\right) + \left(\frac{r_+^6}{l^6}\right) - \left(\frac{r_+^8}{l^8}\right)\right) + \mathcal{O}\left(\frac{r_+^{10}}{l^{10}}\right)\right) \\
a_{1\ 2}^{in}(z) &= \left(\frac{Q^2}{l^4}\right) \left(\frac{6\lambda l^4\Omega_H}{z^2} \left(\frac{l^2}{r_+^2}\right) - \frac{6\lambda l^4\Omega_H(l^2\Omega_H^2+2)}{z^2} + \frac{18\lambda l^4\Omega_H(l^2\Omega_H^2+1)}{z^2} \left(\frac{r_+^2}{l^2}\right) \right. \\
&\quad \left. - \frac{12\lambda l^4\Omega_H(3l^2\Omega_H^2+2)}{z^2} \left(\frac{r_+^4}{l^4}\right) + \mathcal{O}\left(\frac{r_+^6}{l^6}\right)\right) \\
a_{1\ 3}^{in}(z) &= \left(\frac{Q^3}{l^6}\right) \left(-\frac{12\sqrt{3}\lambda^2 l^4\Omega_H}{z^2} \left(\frac{l^4}{r_+^4}\right) + \frac{l^4\Omega_H}{3\sqrt{3}z^6} \left(3(12\lambda^2-1)l^6\Omega_H^2 + 4(12\lambda^2-1)l^4 \right. \right. \\
&\quad \times \Omega_H^2 z^2 + 3(96\lambda^2+1)l^2\Omega_H^2 z^4 + 324\lambda^2 z^4) \left(\frac{l^2}{r_+^2}\right) + \frac{l^2\Omega_H}{6\sqrt{3}z^8} \left(l^2(4(12\lambda^2-1)l^{10} \right. \\
&\quad \times \Omega_H^4 + 3(12\lambda^2-1)l^8\Omega_H^4 z^2 - 4(12\lambda^2-1)l^6\Omega_H^2 z^2(\Omega_H^2 z^2 + 6) - 8l^4\Omega_H^2 z^4(6\lambda^2 \\
&\quad \times (9\Omega_H^2 z^2 + 10) - 5) - 18(132\lambda^2+1)l^2\Omega_H^2 z^6 + 16\Omega_H^2 z^8 - 48\lambda^2 z^6(4\Omega_H^2 z^2 \\
&\quad \left. \left. + 27)\right) - 16(12\lambda^2-1)\Omega_H^2 z^8 \left(z^2 \log\left(1 - \frac{l^2}{z^2}\right) + l^2\right) \log\left(\frac{z^2}{l^2}\right) + 16(12\lambda^2 \right. \\
&\quad \left. - 1)\Omega_H^2 z^{10} \text{Li}_2\left(\frac{l^2}{z^2}\right)\right) + \mathcal{O}\left(\frac{r_+^2}{l^2}\right) \\
a_{1\ 4}^{in}(z) &= \left(\frac{Q^4}{l^8}\right) \left(\frac{72\lambda^3 l^4\Omega_H}{z^2} \left(\frac{l^6}{r_+^6}\right) - \frac{2\lambda l^4\Omega_H}{3z^6} \left(8(12\lambda^2-1)l^6\Omega_H^2 + 12(12\lambda^2-1) \right. \right. \\
&\quad \times l^4\Omega_H^2 z^2 + 3(204\lambda^2+1)l^2\Omega_H^2 z^4 + 432\lambda^2 z^4) \left(\frac{l^4}{r_+^4}\right) + \mathcal{O}\left(\frac{l^2}{r_+^2}\right) \\
&\quad \left. \left. \times \left(\frac{l^2}{r_+^2}\right)\right)\right)
\end{aligned}$$

For the gauge field.

It is straightforward to check that for $\lambda = \lambda_{SG}$ the expansions in section C.2 for the supergravity solution are recovered.

C Expansion of the Supergravity Solution

In this appendix we present for comparison and completeness the far-field and near-field expansions of the exact solution (9).

C.1 Far-field Expansion

The far-field expansion of (9) is

$$\begin{aligned}
f_{SG}^{out}(r) &= 1 + \frac{r^2}{l^2} - \frac{l^2}{r^2} \left(\frac{r_+^2}{l^2} \right) - \frac{l^2 (l^2 \Omega_H^2 + 1)}{r^2} \left(\frac{r_+^4}{l^4} \right) + \frac{l^4 \Omega_H^2}{r^4} (l^2 (1 - \Omega_H^2 r^2) + r^2) \left(\frac{r_+^6}{l^6} \right) \\
&\quad - \frac{l^4 \Omega_H^2}{r^4} (l^2 \Omega_H^2 - 1) (l^2 (\Omega_H^2 r^2 - 1) - r^2) \left(\frac{r_+^8}{l^8} \right) + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) + \left(\frac{Q^2}{l^4} \right) \left(-\frac{l^2}{r^2} \right. \\
&\quad \times \left(\frac{l^2}{r_+^2} \right) + \frac{l^4}{r^4} + \frac{l^6 \Omega_H^4}{r^2} \left(\frac{r_+^2}{l^2} \right) + \frac{l^4 \Omega_H^2}{r^4} (l^2 \Omega_H^2 (l^2 (\Omega_H^2 r^2 - 1) - 4r^2) + r^2) \left(\frac{r_+^4}{l^4} \right) \\
&\quad \left. + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) \right) + \left(\frac{Q^3}{l^6} \right) \left(-\frac{2l^6 \Omega_H^4}{r^2} + \frac{2l^4 \Omega_H^2}{r^4} (l^2 \Omega_H^2 (l^2 + 5r^2) - r^2) \left(\frac{r_+^2}{l^2} \right) \right. \\
&\quad \left. + \mathcal{O} \left(\frac{r_+^4}{l^4} \right) \right) + \left(\frac{Q^4}{l^8} \right) \left(\mathcal{O} \left(\frac{l^2}{r_+^2} \right) \right) \\
g_{SG}^{out}(r) &= 1 - \frac{l^6 \Omega_H^2}{r^4} \left(\frac{r_+^6}{l^6} \right) - \frac{l^6 \Omega_H^2 (l^2 \Omega_H^2 - 1)}{r^4} \left(\frac{r_+^8}{l^8} \right) + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) + \left(\frac{Q^2}{l^4} \right) \left(\frac{l^6 \Omega_H^2}{r^6} (r^2 \right. \\
&\quad \times (l^2 \Omega_H^2 - 1) + l^2) \left(\frac{r_+^4}{l^4} \right) + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) \left. \right) + \left(\frac{Q^3}{l^6} \right) \left(-\frac{2l^6 \Omega_H^2}{r^6} (r^2 (l^2 \Omega_H^2 - 1) \right. \\
&\quad \left. + l^2) \left(\frac{r_+^2}{l^2} \right) + \mathcal{O} \left(\frac{r_+^4}{l^4} \right) \right) + \left(\frac{Q^4}{l^8} \right) \left(\mathcal{O} \left(\frac{l^2}{r_+^2} \right) \right) \\
h_{SG}^{out}(r) &= 1 + \frac{l^6 \Omega_H^2}{r^4} \left(\frac{r_+^6}{l^6} \right) + \frac{l^6 \Omega_H^2 (l^2 \Omega_H^2 - 1)}{r^4} \left(\frac{r_+^8}{l^8} \right) + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) + \left(\frac{Q^2}{l^4} \right) \left(-\frac{l^6 \Omega_H^2}{r^6} (r^2 \right. \\
&\quad \times (l^2 \Omega_H^2 - 1) + l^2) \left(\frac{r_+^4}{l^4} \right) + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) \left. \right) + \left(\frac{Q^3}{l^6} \right) \left(\frac{2l^6 \Omega_H^2}{r^6} (r^2 (l^2 \Omega_H^2 - 1) + l^2) \right. \\
&\quad \times \left(\frac{r_+^2}{l^2} \right) + \mathcal{O} \left(\frac{r_+^4}{l^4} \right) \left. \right) + \left(\frac{Q^4}{l^8} \right) \left(\mathcal{O} \left(\frac{l^2}{r_+^2} \right) \right) \\
\Omega_{SG}^{out}(r) &= \frac{l^4 \Omega_H}{r^4} \left(\frac{r_+^4}{l^4} \right) + \frac{l^6 \Omega_H^3}{r^4} \left(\frac{r_+^6}{l^6} \right) + \frac{l^6 \Omega_H^3 (l^2 \Omega_H^2 - 1)}{r^4} \left(\frac{r_+^8}{l^8} \right) + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) + \left(\frac{Q^2}{l^4} \right) \\
&\quad \times \left(\frac{l^4 \Omega_H}{r^4} - \frac{l^4 \Omega_H}{r^6} (l^2 (\Omega_H^2 r^2 + 1) + r^2) \left(\frac{r_+^2}{l^2} \right) + \frac{l^4 \Omega_H}{r^6} (r^2 (1 - l^2 \Omega_H^2 (l^2 \Omega_H^2 - 3)) \right. \\
&\quad \left. + l^2) \left(\frac{r_+^4}{l^4} \right) + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) \right) + \left(\frac{Q^3}{l^6} \right) \left(-\frac{l^4 \Omega_H}{r^4} \left(\frac{l^2}{r_+^2} \right) + \frac{l^4 \Omega_H}{r^6} (r^2 (3l^2 \Omega_H^2 + 2) \right. \\
&\quad \left. + l^2) - \frac{l^4 \Omega_H}{r^6} (l^2 (\Omega_H^2 (l^2 + 11r^2) + 2) + 3r^2) \left(\frac{r_+^2}{l^2} \right) + \mathcal{O} \left(\frac{r_+^4}{l^4} \right) \right) + \left(\frac{Q^4}{l^8} \right) \\
&\quad \times \left(\frac{l^4 \Omega_H}{r^4} \left(\frac{l^4}{r_+^4} \right) + \mathcal{O} \left(\frac{l^2}{r_+^2} \right) \right)
\end{aligned}$$

for the metric functions. For the gauge field we obtain

$$\begin{aligned}
a_0^{out}{}_{SG}(r) &= \left(\frac{Q}{l^2}\right) \frac{\sqrt{3}l^2}{r^2} \\
a_1^{out}{}_{SG} &= \left(\frac{Q}{l^2}\right) \left(\frac{\sqrt{3}l^4\Omega_H}{r^2} \left(-\left(\frac{r_+^2}{l^2}\right) + \left(\frac{r_+^4}{l^4}\right) - \left(\frac{r_+^6}{l^6}\right) + \left(\frac{r_+^8}{l^8}\right) \right) + \mathcal{O}\left(\frac{r_+^{10}}{l^{10}}\right) \right) \\
&\quad + \left(\frac{Q^2}{l^4}\right) \left(\frac{\sqrt{3}l^4\Omega_H}{r^2} - \frac{\sqrt{3}l^4\Omega_H(l^2\Omega_H^2+2)}{r^2} \left(\frac{r_+^2}{l^2}\right) + \frac{3\sqrt{3}l^4\Omega_H(l^2\Omega_H^2+1)}{r^2} \right. \\
&\quad \times \left(\frac{r_+^4}{l^4}\right) + \mathcal{O}\left(\frac{r_+^6}{l^6}\right) \Big) + \left(\frac{Q^3}{l^6}\right) \left(-\frac{\sqrt{3}l^4\Omega_H}{r^2} \left(\frac{l^2}{r_+^2}\right) + \frac{3\sqrt{3}l^4\Omega_H(l^2\Omega_H^2+1)}{r^2} \right. \\
&\quad \left. - \frac{2\sqrt{3}l^4\Omega_H}{r^2} (l^2\Omega_H^2(l^2\Omega_H^2+6)+3) \left(\frac{r_+^2}{l^2}\right) + \mathcal{O}\left(\frac{r_+^4}{l^4}\right) \right) + \left(\frac{Q^4}{l^8}\right) \left(\frac{\sqrt{3}l^4\Omega_H}{r^2} \right. \\
&\quad \left. \times \left(\frac{l^4}{r_+^4}\right) + \mathcal{O}\left(\frac{l^2}{r_+^2}\right) \right)
\end{aligned}$$

C.2 Near-field Expansion

The near-field expansion of (9) is considerably lengthier, and so we present the results for the metric functions in successive powers of Q/l^2 . The index represents the order of Q/l^2 . We find

$$\begin{aligned}
f_{SG0}^{in}(z) &= 1 - \frac{l^2}{z^2} + \frac{1}{l^2 z^4} (l^8\Omega_H^2 - l^4 z^2 (l^2\Omega_H^2 + 1) + z^6) \left(\frac{r_+^2}{l^2}\right) + \frac{l^4\Omega_H^2}{z^4} (l^2\Omega_H^2 - 1) \\
&\quad \times (l^2 - z^2) \left(\frac{r_+^4}{l^4}\right) + \frac{l^4\Omega_H^2}{z^4} (l^2\Omega_H^2 - 1)^2 (l^2 - z^2) \left(\frac{r_+^6}{l^6}\right) + \frac{l^4\Omega_H^2}{z^4} (l^2\Omega_H^2 - 1)^3 \\
&\quad \times (l^2 - z^2) \left(\frac{r_+^8}{l^8}\right) + \mathcal{O}\left(\frac{r_+^{10}}{l^{10}}\right) \\
f_{SG2}^{in}(z) &= \left(\frac{Q^2}{l^4}\right) \left(\frac{l^2(l^2 - z^2)}{z^4} \left(\frac{l^4}{r_+^4}\right) - \frac{l^6\Omega_H^4(l^2 - z^2)}{z^4} - \frac{l^4\Omega_H^2}{z^4} (l^2\Omega_H^2 (l^2\Omega_H^2 - 4) + 1) \right. \\
&\quad \left. (l^2 - z^2) \left(\frac{r_+^2}{l^2}\right) - \frac{l^4\Omega_H^2}{z^4} (l^2\Omega_H^2 (l^2\Omega_H^2 (l^2\Omega_H^2 - 5) + 10) - 3) (l^2 - z^2) \left(\frac{r_+^4}{l^4}\right) \right. \\
&\quad \left. + \mathcal{O}\left(\frac{r_+^6}{l^6}\right) \right) \\
f_{SG3}^{in}(z) &= \left(\frac{Q^3}{l^6}\right) \left(\frac{2l^6\Omega_H^4}{z^4} (l^2 - z^2) \left(\frac{l^2}{r_+^2}\right) + \frac{2l^4\Omega_H^2}{z^4} (1 - 5l^2\Omega_H^2) (l^2 - z^2) + \frac{2l^4\Omega_H^2}{z^4} \right. \\
&\quad \left. \times (14l^2\Omega_H^2 - 4) (l^2 - z^2) \left(\frac{r_+^2}{l^2}\right) + \mathcal{O}\left(\frac{r_+^4}{l^4}\right) \right)
\end{aligned}$$

$$\begin{aligned}
f_{SG4}^{in}(z) &= \left(\frac{Q^4}{l^8} \right) \left(\frac{3l^6\Omega_H^4(z^2 - l^2)}{z^4} \left(\frac{l^4}{r_+^4} \right) + \mathcal{O} \left(\frac{l^2}{r_+^2} \right) \right) \quad (31) \\
g_{SG0}^{in}(z) &= 1 - \frac{l^6\Omega_H^2}{z^4} \left(\frac{r_+^2}{l^2} \right) + \frac{l^6\Omega_H^2}{z^8} (l^2\Omega_H^2(l^4 - z^4) + z^4) \left(\frac{r_+^4}{l^4} \right) - \frac{l^6\Omega_H^2}{z^{12}} (l^2\Omega_H^2(l^4 - z^4) \\
&\quad + z^4)^2 \left(\frac{r_+^6}{l^6} \right) + \frac{l^6\Omega_H^2}{z^{16}} (l^2\Omega_H^2(l^4 - z^4) + z^4)^3 \left(\frac{r_+^8}{l^8} \right) + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) \\
g_{SG2}^{in}(z) &= \left(\frac{Q^2}{l^4} \right) \left(\frac{l^8\Omega_H^2}{z^6} \left(\frac{l^2}{r_+^2} \right) - \frac{l^6\Omega_H^2}{z^{10}} (2l^8\Omega_H^2 + l^2z^4(2 - \Omega_H^2z^2) + z^6) + \frac{l^6\Omega_H^2}{z^{14}} (3l^{14}\Omega_H^4 \right. \\
&\quad + z^{10}(l^2\Omega_H^2(l^2\Omega_H^2 - 4) + 3) + 3l^2z^8 - 2l^8\Omega_H^2z^4(l^2\Omega_H^2 - 3) - 2l^6\Omega_H^2z^6(l^2\Omega_H^2 \\
&\quad - 1)) \left(\frac{r_+^2}{l^2} \right) + \frac{l^6\Omega_H^2}{z^{18}} (-4l^{20}\Omega_H^6 - 4l^2z^{12} + 6l^{14}\Omega_H^4z^4(l^2\Omega_H^2 - 2) + 3l^{12}\Omega_H^4z^6 \\
&\quad \times (l^2\Omega_H^2 - 1) - 2l^8\Omega_H^2z^8(l^2\Omega_H^2(l^2\Omega_H^2 - 4) + 6) - 4l^6\Omega_H^2z^{10}(l^2\Omega_H^2(l^2\Omega_H^2 - 3) \\
&\quad + 2) + z^{14}(l^2\Omega_H^2(l^4\Omega_H^4 - 5l^2\Omega_H^2 + 10) - 6)) \left(\frac{r_+^4}{l^4} \right) + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) \Big) \\
g_{SG3}^{in}(z) &= \left(\frac{Q^3}{l^6} \right) \left(-\frac{2l^8\Omega_H^2}{z^6} \left(\frac{l^4}{r_+^4} \right) + \frac{2l^6\Omega_H^2}{z^{10}} (l^4\Omega_H^2(2l^4 + z^4) + l^2z^4(3 - \Omega_H^2z^2) + z^6) \right. \\
&\quad \times \left(\frac{l^2}{r_+^2} \right) - \frac{2l^6\Omega_H^2}{z^{14}} (-4z^{10}(l^2\Omega_H^2 - 1) + 2l^2z^8(2l^2\Omega_H^2 + 3) + l^8\Omega_H^2(3l^6\Omega_H^2 \\
&\quad + 8z^4) - 2l^6\Omega_H^2z^6(l^2\Omega_H^2 - 1)) + \frac{2l^6\Omega_H^2}{z^{18}} (4l^{20}\Omega_H^6 + 20l^8\Omega_H^2z^8 - 10z^{14}(l^2\Omega_H^2 \\
&\quad - 1) + 10l^2z^{12}(l^2\Omega_H^2 + 1) - 3l^{14}\Omega_H^4z^4(l^2\Omega_H^2 - 5) - 3l^{12}\Omega_H^4z^6(l^2\Omega_H^2 - 1) \\
&\quad + 2l^6\Omega_H^2z^{10}(l^2\Omega_H^2(l^2\Omega_H^2 - 6) + 5)) \left(\frac{r_+^2}{l^2} \right) + \mathcal{O} \left(\frac{r_+^4}{l^4} \right) \Big) \\
g_{SG4}^{in}(z) &= \left(\frac{Q^4}{l^8} \right) \left(\frac{3l^8\Omega_H^2}{z^6} \left(\frac{l^6}{r_+^6} \right) + \frac{l^6\Omega_H^2}{z^{12}} (3z^8(l^2\Omega_H^2 - 1) - 4l^2z^6(2l^2\Omega_H^2 + 3) + l^8\Omega_H^2 \right. \\
&\quad \times (l^2 - 6z^2)) \left(\frac{l^4}{r_+^4} \right) + \mathcal{O} \left(\frac{l^2}{r_+^2} \right) \Big) \\
h_{SG0}^{in}(z) &= 1 + \frac{l^6\Omega_H^2}{z^4} \left(\frac{r_+^2}{l^2} \right) + \frac{l^6\Omega_H^2}{z^4} (l^2\Omega_H^2 - 1) \left(\frac{r_+^4}{l^4} \right) + \frac{l^6\Omega_H^2}{z^4} (l^2\Omega_H^2 - 1)^2 \left(\frac{r_+^6}{l^6} \right) \\
&\quad + \frac{l^6\Omega_H^2}{z^4} (l^2\Omega_H^2 - 1)^3 \left(\frac{r_+^8}{l^8} \right) + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) \\
h_{SG2}^{in}(z) &= \left(\frac{Q^2}{l^4} \right) \left(-\frac{l^8\Omega_H^2}{z^6} \left(\frac{l^2}{r_+^2} \right) + \frac{l^6\Omega_H^2}{z^6} (l^2(2 - \Omega_H^2z^2) + z^2) - \frac{l^6\Omega_H^2}{z^6} (z^2(l^2\Omega_H^2 \right. \\
&\quad \times (l^2\Omega_H^2 - 4) + 3) + 3l^2) \left(\frac{r_+^2}{l^2} \right) + \frac{l^6\Omega_H^2}{z^6} (z^2(6 - l^2\Omega_H^2(l^2\Omega_H^2(l^2\Omega_H^2 - 5) \\
&\quad + 10)) + 4l^2) \left(\frac{r_+^4}{l^4} \right) + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) \Big)
\end{aligned}$$

$$\begin{aligned}
h_{SG3}^{in}(z) &= \left(\frac{Q^3}{l^6}\right) \left(\frac{2l^8\Omega_H^2}{z^6} \left(\frac{l^4}{r_+^4}\right) - \frac{2l^6\Omega_H^2}{z^6} (l^2 (\Omega_H^2 (l^2 - z^2) + 3) + z^2) \left(\frac{l^2}{r_+^2}\right) + \frac{4l^6\Omega_H^2}{z^6}\right. \\
&\quad \times (l^2 (2\Omega_H^2 (l^2 - z^2) + 3) + 2z^2) - \frac{20l^6\Omega_H^2}{z^6} (l^2 (\Omega_H^2 (l^2 - z^2) + 1) + z^2) \left(\frac{r_+^2}{l^2}\right) \\
&\quad \left. + \mathcal{O}\left(\frac{r_+^4}{l^4}\right)\right) \\
h_{SG4}^{in}(z) &= \left(\frac{Q^4}{l^8}\right) \left(-\frac{3l^8\Omega_H^2}{z^6} \left(\frac{l^6}{r_+^6}\right) + \frac{l^6\Omega_H^2}{z^6} (l^2 (\Omega_H^2 (8l^2 - 3z^2) + 12) + 3z^2) \left(\frac{l^4}{r_+^4}\right)\right. \\
&\quad \left. + \mathcal{O}\left(\frac{l^2}{r_+^2}\right)\right) \\
\Omega_{SG0}^{in}(z) &= \frac{l^4\Omega_H}{z^4} - \frac{l^6\Omega_H^3}{z^8} (l^4 - z^4) \left(\frac{r_+^2}{l^2}\right) + \frac{l^6\Omega_H^3}{z^{12}} (l^4 - z^4) (l^2\Omega_H^2 (l^4 - z^4) + z^4) \left(\frac{r_+^4}{l^4}\right) \\
&\quad - \frac{l^6\Omega_H^3}{z^{16}} (l^4 - z^4) (l^2\Omega_H^2 (l^4 - z^4) + z^4)^2 \left(\frac{r_+^6}{l^6}\right) + \frac{l^6\Omega_H^3}{z^{20}} (l^4 - z^4) (l^2\Omega_H^2 \\
&\quad \times (l^4 - z^4) + z^4)^3 \left(\frac{r_+^8}{l^8}\right) + \mathcal{O}\left(\frac{r_+^{10}}{l^{10}}\right) \\
\Omega_{SG2}^{in}(z) &= \left(\frac{Q^2}{l^4}\right) \left(-\frac{l^4\Omega_H}{z^6} (l^2 - z^2) \left(\frac{l^4}{r_+^4}\right) + \frac{l^4\Omega_H}{z^{10}} (l^2 - z^2) (l^2\Omega_H^2 (2l^4 + l^2z^2 + z^4) \right. \\
&\quad \left. + z^4) \left(\frac{l^2}{r_+^2}\right) - \frac{l^4\Omega_H}{z^{14}} (l^2 - z^2) \left(z^8 (1 - l^2\Omega_H^2 (l^2\Omega_H^2 - 3)) + l^4\Omega_H^2 z^6 (3 - l^2\Omega_H^2) \right. \right. \\
&\quad \left. \left. + l^6\Omega_H^2 (l^4\Omega_H^2 (3l^2 + 2z^2) + 4z^4)\right) + \frac{l^4\Omega_H}{z^{18}} \left(-z^{14} (l^2\Omega_H^2 (l^2\Omega_H^2 (l\Omega_H - 2) (l\Omega_H \right. \right. \\
&\quad \left. \left. + 2) + 6) + 1) + l^2z^{12} + l^{18}\Omega_H^6 (4l^2 - z^2) + 3l^{14}\Omega_H^4 z^4 (3 - 2l^2\Omega_H^2) - l^{12}\Omega_H^4 z^6 \right. \right. \\
&\quad \left. \left. \times (l^2\Omega_H^2 + 1) + 2l^8\Omega_H^2 z^8 (l^2\Omega_H^2 (l^2\Omega_H^2 - 3) + 3) + 3l^8\Omega_H^4 z^{10} (l^2\Omega_H^2 - 2) \right) \right. \\
&\quad \left. \times \left(\frac{r_+^2}{l^2}\right) + \frac{l^4\Omega_H}{z^{22}} \left(-l^2z^{16} + l^{24}\Omega_H^8 (z^2 - 5l^2) + 4l^{20}\Omega_H^6 z^4 (3l^2\Omega_H^2 - 4) + l^{18} \right. \right. \\
&\quad \left. \left. \times \Omega_H^6 z^6 (l^2\Omega_H^2 + 1) - 3l^{14}\Omega_H^4 z^8 (l^2\Omega_H^2 (3l^2\Omega_H^2 - 8) + 6) + l^{12}\Omega_H^4 z^{10} (-l^2\Omega_H^2 \right. \right. \\
&\quad \left. \left. \times (6l^2\Omega_H^2 - 11) - 2) + 2l^8\Omega_H^2 z^{12} (l^2\Omega_H^2 (l^2\Omega_H^2 (l\Omega_H - 2) (l\Omega_H + 2) + 6) - 4) \right. \right. \\
&\quad \left. \left. + l^6\Omega_H^2 z^{14} (l^2\Omega_H^2 (l^2\Omega_H^2 (5l^2\Omega_H^2 - 17) + 18) - 2) + z^{18} (1 - l^2\Omega_H^2 (l^6\Omega_H^6 - 5l^4 \right. \right. \\
&\quad \left. \left. \times \Omega_H^4 + 10l^2\Omega_H^2 - 10))\right) \right) \left(\frac{r_+^4}{l^4}\right) + \mathcal{O}\left(\frac{r_+^6}{l^6}\right) \\
\Omega_{SG3}^{in}(z) &= \left(\frac{Q^3}{l^6}\right) \left(\frac{l^4\Omega_H}{z^6} (l^2 - z^2) \left(\frac{l^6}{r_+^6}\right) - \frac{l^4\Omega_H}{z^{10}} (l^2 - z^2) \left(z^4 (3l^2\Omega_H^2 + 2) + l^4\Omega_H^2 (3l^2 \right. \right. \\
&\quad \left. \left. + 2z^2)\right) \left(\frac{l^4}{r_+^4}\right) + \frac{l^4\Omega_H}{z^{14}} (l^2 - z^2) \left(8l^4\Omega_H^2 z^6 + z^8 (11l^2\Omega_H^2 + 3) + l^{10}\Omega_H^4 (5l^2 \right. \right. \\
&\quad \left. \left. + 4z^2) + l^6\Omega_H^2 z^4 (4l^2\Omega_H^2 + 9)\right) \right) \left(\frac{l^2}{r_+^2}\right) - \frac{l^4\Omega_H}{z^{18}} \left(-2z^{14} (13l^2\Omega_H^2 + 2) + 2l^2z^{12} \right.
\end{aligned}$$

$$\begin{aligned}
& \times (3l^2\Omega_H^2 + 2) - 5l^{14}\Omega_H^4 z^4 (l^2\Omega_H^2 - 4) + 2l^6\Omega_H^2 z^{10} (2l^2\Omega_H^2 (l^2\Omega_H^2 - 5) + 1) \\
& + l^8\Omega_H^2 (l^6\Omega_H^4 (7l^6 - l^4 z^2 - 5z^6) + 18z^8) + \frac{l^4\Omega_H}{z^{22}} (l^2 - z^2) (40l^4\Omega_H^2 z^{14} + 5z^{16} \\
& \times (10l^2\Omega_H^2 + 1) + l^{22}\Omega_H^8 (9l^2 + 8z^2) + l^{18}\Omega_H^6 z^4 (35 - 6l^2\Omega_H^2) - 12l^{16}\Omega_H^6 z^6 \\
& \times (l^2\Omega_H^2 - 3) + l^{12}\Omega_H^4 z^8 (50 - l^2\Omega_H^2 (7l^2\Omega_H^2 - 11)) + 4l^{10}\Omega_H^4 z^{10} (l^2\Omega_H^2 (l^2\Omega_H^2 \\
& - 6) + 15) + 2l^6\Omega_H^2 z^{12} (2l^2\Omega_H^2 (l^2\Omega_H^2 (l^2\Omega_H^2 - 6) + 15) + 15)) \left(\frac{r_+^2}{l^2} \right) + \mathcal{O} \left(\frac{r_+^4}{l^4} \right) \Big) \\
\Omega_{SG4}^{in}(z) = & \left(\frac{Q^4}{l^8} \right) \left(\frac{l^4\Omega_H}{z^6} (z^2 - l^2) \left(\frac{l^8}{r_+^8} \right) - \frac{l^4\Omega_H}{z^{12}} (l^2 - z^2) \left(l^4\Omega_H^2 (l^4 - 4l^2 z^2 - 3z^4) \right. \right. \\
& \left. \left. - 3z^6 (2l^2\Omega_H^2 + 1) \right) \left(\frac{l^6}{r_+^6} \right) + \frac{l^4\Omega_H}{z^{16}} \left(z^{12} (l^2\Omega_H^2 (7l^2\Omega_H^2 + 27) + 6) - 2l^2 z^{10} \right. \right. \\
& \left. \left. \times (l^2\Omega_H^2 (l^2\Omega_H^2 + 6) + 3) + l^{10}\Omega_H^2 z^4 (l^2\Omega_H^2 + 3) + 3l^8\Omega_H^4 (l^8 - 3l^6 z^2 + 3z^8) \right. \right. \\
& \left. \left. - 9l^8\Omega_H^2 z^6 (l^2\Omega_H^2 + 2) \right) \left(\frac{l^4}{r_+^4} \right) + \mathcal{O} \left(\frac{l^2}{r_+^2} \right) \right)
\end{aligned}$$

for the metric functions. For the gauge field we find

$$\begin{aligned}
a_{0\ SG}^{in}(z) &= \left(\frac{Q}{l^2} \right) \frac{\sqrt{3}l^2}{z^2} \left(\frac{l}{r_+} \right) \\
a_{1\ SG1}^{in}(z) &= \left(\frac{Q}{l^2} \right) \left(-\frac{\sqrt{3}l^4\Omega_H}{z^2} \left(1 - \left(\frac{r_+^2}{l^2} \right) + \left(\frac{r_+^4}{l^4} \right) - \left(\frac{r_+^6}{l^6} \right) + \left(\frac{r_+^8}{l^8} \right) \right) + \mathcal{O} \left(\frac{r_+^{10}}{l^{10}} \right) \right) \\
a_{1\ SG2}^{in}(z) &= \left(\frac{Q^2}{l^4} \right) \left(\frac{\sqrt{3}l^4\Omega_H}{z^2} \left(\frac{l^2}{r_+^2} \right) - \frac{\sqrt{3}l^4\Omega_H (l^2\Omega_H^2 + 2)}{z^2} + \frac{3\sqrt{3}l^4\Omega_H (l^2\Omega_H^2 + 1)}{z^2} \left(\frac{r_+^2}{l^2} \right) \right. \\
& \left. - \frac{2\sqrt{3}l^4\Omega_H (3l^2\Omega_H^2 + 2)}{z^2} \left(\frac{r_+^4}{l^4} \right) + \mathcal{O} \left(\frac{r_+^6}{l^6} \right) \right) \\
a_{1\ SG3}^{in}(z) &= \left(\frac{Q^3}{l^6} \right) \left(-\frac{\sqrt{3}l^4\Omega_H}{z^2} \left(\frac{l^4}{r_+^4} \right) + \frac{3\sqrt{3}l^4\Omega_H (l^2\Omega_H^2 + 1)}{z^2} \left(\frac{l^2}{r_+^2} \right) - \frac{2\sqrt{3}l^4\Omega_H}{z^2} \right. \\
& \left. \times (l^2\Omega_H^2 (l^2\Omega_H^2 + 6) + 3) + \mathcal{O} \left(\frac{r_+^2}{l^2} \right) \right) \\
a_{1\ SG4}^{in}(z) &= \left(\frac{Q^4}{l^8} \right) \left(\frac{\sqrt{3}l^4\Omega_H}{z^2} \left(\frac{l^6}{r_+^6} \right) - \frac{2\sqrt{3}l^4\Omega_H (3l^2\Omega_H^2 + 2)}{z^2} \left(\frac{l^4}{r_+^4} \right) + \mathcal{O} \left(\frac{l^2}{r_+^2} \right) \right)
\end{aligned}$$

References

- [1] L. Susskind and E. Witten, “*The Holographic Bound in Anti-de Sitter Space*”, [arXiv:hep-th/9805114].

- [2] M. Mir, “On Holographic Weyl Anomaly”, JHEP **1310**, 084 (2013) [arXiv:1307.5514v3 [hep-th]]
- [3] F. R. Tangherlini, Il Nuovo Cimento **27**, 636 (1963)
- [4] R. C. Myers and M. J. Perry, Ann. Phys. (N.Y.) **172**, 304 (1986).
- [5] G. W. Gibbons, H. Lu, D. N. Page and C. N. Pope, “*Rotating Black Holes in Higher Dimensions with a Cosmological Constant*”, Phys. Rev. Lett. **93**, 171102 (2004) [arXiv:hep-th/0409155].
- [6] Z-W. Chong, M. Cvetič, H. Lu and C. N. Pope, “*General Non-Extremal Rotating Black Holes in Minimal Five-Dimensional Gauged Supergravity*”, Phys. Rev. Lett. **95**, 161301 (2005) [arXiv:hep-th/0506029].
- [7] J. Kunz, F. Navarro-Lerida and E. Radu, “*Higher dimensional rotating black holes in Einstein-Maxwell theory with negative cosmological constant*”, Phys. Lett. B **649**, 463 (2007) [arXiv:gr-qc/0702086].
- [8] Y. Brihaye and T. Delsate, “*Charged-Rotating Black Holes in Higher-dimensional (A)DS-Gravity*”, Phys. Rev. D **79**, 105013 (2009) [arXiv:0806.1583 [gr-qc]].
- [9] H. C. Kim and R. G. Cai, “*Slowly Rotating Charged Gauss-Bonnet Black holes in AdS Spaces*”, Phys. Rev. D **77**, 024045 (2008) [arXiv:0711.0885 [hep-th]].
- [10] J. Kunz and F. Navarro-Lerida, “*D=5 Einstein-Maxwell-Chern-Simons Black Holes*”, Phys. Rev. Lett. **96**, 081101 (2006) [arXiv:hep-th/0510250].
- [11] J. Kunz and F. Navarro-Lerida, “*Negative Horizon Mass for Rotating Black Holes*”, Phys. Lett. B **643**, 55 (2006) [arXiv:hep-th/0610036].
- [12] J. C. Breckenridge, R. C. Myers, A. W. Peet, C. Vafa, Phys. Lett. B **391** (1997) 93.
- [13] J. P. Gauntlett, R. C. Myers, and P. K. Townsend, Class. Quantum Grav. **16** (1999) 1.
- [14] C. A. R. Herdeiro, Nucl. Phys. B **582** (2000) 363; P. K. Townsend, Annales Henri Poincaré **4** (2003) 183.
- [15] B. Kleihaus, J. Kunz, and F. Navarro-Lerida, “*Stationary Black Holes with Static and Counterrotating Horizons*”, Phys. Rev. D **69** (2004) 081501 [arXiv:gr-qc/0309082].
- [16] J. L. Blazquez-Salcedo, J. Kunz, F. Navarro-Lerida and E. Radu, “*Sequences of extremal radially excited rotating black holes*”, Phys. Rev. Lett. **112**, 011101 (2014) [arXiv:1308.0548 [gr-qc]].

- [17] J. L. Blzquez-Salcedo, J. Kunz, F. Navarro-Lrida, E. Radu, “*Radially excited rotating black holes in Einstein-Maxwell-Chern-Simons theory*”, Phys. Rev. D **92**, 044025 (2015) [arXiv:1506.07802 [gr-qc]].
- [18] A. N. Aliev, “*Gyromagnetic Ratio of Charged Kerr-Anti-de Sitter Black Holes*”, Class.Quant.Grav. **24**:4669-4678,2007 [arXiv:hep-th/0611205].
- [19] A. N. Aliev, “*A Slowly Rotating Charged Black Hole in Five Dimensions*”, Mod. Phys. Lett. A **21**, 751 (2006) [arXiv:gr-qc/0505003].
- [20] A. N. Aliev, “*Rotating Black Holes in Higher Dimensional Einstein-Maxwell Gravity*”, Phys. Rev. D **74**, 024011 (2006) [arXiv:hep-th/0604207].
- [21] F. Navarro-Lerida, “*Perturbative Charged Rotating 5D Einstein-Maxwell Black Holes*”, Gen. Rel. Grav. **42**:2891-2900, 2010 [arXiv:0706.0591 [hep-th]].
- [22] M. Allahverdizadeh, J. Kunz, F. Navarro-Lerida, “*Extremal Charged Rotating Black Holes in Odd Dimensions*”, Phys.Rev. D **82** (2010) 024030 [arXiv:1004.5050 [gr-qc]].
- [23] W. Israel, Commun. Math. Phys. **8**, 245 (1968).
- [24] D. C. Robinson, Phys. Rev. Lett. **34**, 905 (1975).
- [25] P. O. Mazur, J. Phys. A **15**, 3173 (1982).
- [26] P. T. Chrusciel, J. L. Costa and M. Heusler, “*Stationary Black Holes: Uniqueness and Beyond*”, Living Rev. Rel. **15**, 7 (2012) [arXiv:1205.6112 [gr-qc]].
- [27] O. J. C. Dias, G. T. Horowitz, J. E. Santos, “*Black holes with only one Killing field*”, JHEP **1107**, 115 (2011) [arXiv:1105.4167 [hep-th]].
- [28] S. Stotyn, M. Park, P. McGrath and R. B. Mann, “*Black Holes and Boson Stars with One Killing Field in Arbitrary Odd Dimensions*”, Phys. Rev. D **85**, 044036 (2012) [arXiv:1110.2223 [hep-th]].
- [29] M. Allahverdizadeh, J. Kunz, F. Navarro-Lerida, “*Extremal Charged Rotating Dilaton Black Holes in Odd Dimensions*”, Phys.Rev. D **82** (2010) 064034 [arXiv:1007.4250 [gr-qc]].
- [30] P. Basu, J. Bhattacharya, S. Bhattacharyya, R. Loganayagam, S. Minwalla, V. Umesh, “*Small Hairy Black Holes in Global AdS Spacetime*”, JHEP **1010** (2010) 045 [arXiv:1003.3232 [hep-th]].
- [31] A. Ashtekar, S. Das, “*Asymptotically Anti-de Sitter space-times: Conserved quantities*”, Class. Quant. Grav. **17**, L17-L30 (2000) [arXiv:hep-th/9911230].

- [32] S. Das and R. B. Mann, “*Conserved quantities in Kerr-anti-de Sitter space-times in various dimensions*”, JHEP **0008**, 033 (2000) [hep-th/0008028].
- [33] O. Brodbeck, M. Heusler, N. Straumann, and M. S. Volkov, Phys. Rev. Lett. **79** (1997) 4310.
- [34] J.L. Blazquez-Salcedo, J. Kunz, F. Navarro-Lerida, and E. Radu “*Charged rotating black holes in Einstein-Maxwell-Chern-Simons theory with negative cosmological constant*”, [arXiv:1610.05282 [gr-qc]].